Bayes' Theorem - Critical Reasoning - PHI 169

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A. Conditional Probability

The conditional probability of A given B—in symbols, $P(A|B)$—expresses the probability of A *on the assumption that* B holds. The definition of conditional probability is as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

For our proposes, the above definition is equivalent to the following:

$$P(A|B) = \frac{\textit{\# elements in } A \cap B}{\textit{\# elements in } B}.$$

I leave it to you to figure out why the two definitions are equivalent.

Illustration Suppose you throw fair die that has six faces, numbered 'One' - 'Two' - 'Three' - 'Four' - 'Five' - 'Six'. The probability of getting a 'Six' is 1/6, or in symbols $P(Six) = 1/6$ because there six possible outcomes and 'Six' is one of them. What is the probability of getting a 'Six' *conditional on* getting an even number, or in symbols, $Pr(Six|Even)$? We have:

$$P(Six|Even) = \frac{P(Six \cap Even)}{P(Even)} = \frac{\textit{\# elements in } Six \cap Even}{\textit{\# elements in } Even}.$$

We know that $P(Even) = 3/6$ because there are three even numbers one could get while tossing a coin. We also know that $P(Six \cap Even) = 1/6$ because there is only one number that is 'Six' and even, and there are a total of six numbers. Hence,

$$P(Six|Even) = \frac{1/6}{3/6} = 1/3.$$

B. Bayes' Theorem

Bayes' theorem is as follows:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}.$$

Bayes' theorem allows us to calculate the *conditional* probability of A given B from:

(i) the probability $P(A)$ regardless of B

(ii) the probability of B given A, i.e. $P(B|A)$

(iii) the probability of $P(B)$, where $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$

One should not confuse $P(A|B)$ and $P(B|A)$. Bayes' theorem shows how the two are related. At first blush, this theorem might look complicated and hard to understand. We shall thus look at an illustrative example and use it to understand how the theorem works.

C. An illustration: Diagnostic tests

Suppose that a disease is found in 1/100 people in the US. We select a US individual at random and then give the individual a diagnostic test, which is 90% reliable. This means that if an individual has the disease, the probability that the test comes out positive is 90%, and if an individual does not have the disease, the probability that the test comes out negative is again 90%. Suppose the test comes out positive. What is the probability that the individual has the disease, given the positive test result? How does this probability compare with the probability that the individual does *not* have the disease, given that the test result was positive?

Let's first understand what the problem is saying. We are told that an individual, picked at random from the US population, has a probability of 1/100 of having the disease, that is,

$$P(D) = 1/100 = 1\%,$$

where D means "the individual picked at random from the US population has the disease". This 1% is also called the *base rate* for the disease in question.

We are also told that the test is 90% reliable, that is, if the individual has the disease, the test will come out positive in 90% of the cases and if the individual does not have the disease, the test will come out negative in 90% of the cases. In other words, the test result is correct 90% of the time, i.e.

$$P(T[positive]|D) = 90\% \text{ and } P(T[negative]|D^c) = 90\%,$$

where $T[positive]$ means that the test is positive and $T[negative]$ the test is negative. A consequence of this is that the test result is incorrect 10% of the time, i.e.

$$P(T[negative]|D) = 10\% \text{ and } P(T[positive]|D^c) = 10\%,$$

because in general $P(T[negative]) = 1 - P(T[positive])$.

So, the problem is asking us to calculate two probabilities and compare them: first, the probability of having the disease given the that the test comes out positive, i.e. $P(D|T[positive])$; second, the probability of *not* having the disease given that the test comes out positive, i.e. $P(D^c|T[positive])$. Which one of the two is higher?

Let's try to solve the problem intuitively. Suppose an individual tested positive. How likely is it that the individual has the disease? Many answer that there is a 90% probability that the individual has the disease because the tests is correct 90% of the time. *This answer is not correct.*

I. Method of counting cases To arrive at the correct answer, let's picture an imaginary poll of people being tested, say 1,000,000 individuals. Since we are told $P(D) = 1\%$, we know that regardless of the results of the test only 1% of them have the disease. So we know that only 10,000 individuals have the disease regardless of what the test says. The remaining 990,000 do not have the disease. Let's now suppose all 1,000,000 get tested with the test having 90% reliability. What are the possible outcomes? The individuals can test either positive or negative.

First, consider the individuals who tested positive. There are two sub-cases:

- The first sub-case is that the individuals testing positive do in fact have the disease. How many such cases will there be? From our poll of 10,000 who have the disease, 9,000 of them will test positive since the test is 90% reliable.
- The second sub-case is that the individuals testing positive do not in fact have the disease. How many such cases will there be? From our poll of 990,000 who don't have the disease, 99,000 will test positive since the test is 90% reliable (and thus wrong 10% of the time).

Second, consider the individuals who tested negative. There are two sub-cases:

- The first sub-case is that the individuals testing negative do not have the disease. How many such cases will there be? From our poll of 990,000 who do not have the disease, 891,000 of them will test negative since the test is 90% reliable.
- The second sub-case is that the individuals testing negative do in fact have the disease. How many such cases will there be? From our poll of 10,000 who have the disease, 1,000 will test positive since the test is 90% reliable (and thus wrong 10% of the time).

More schematically:

Individuals who test positive

- having the disease: 9,000
- without having the disease: 99,000

Individuals who test negative

- without having the disease: 891,000
- having the disease: 1,000

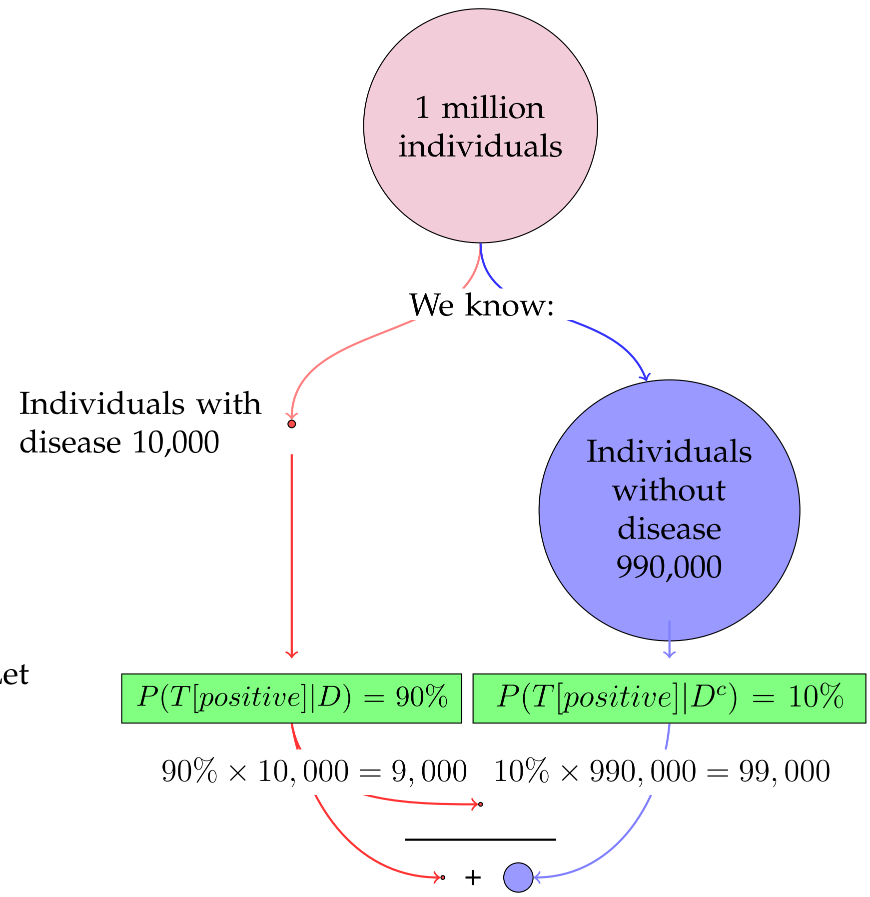
Individuals with disease: 9,000 + 1,000 = 10,000

Individuals without disease: 99,000 + 891,000 = 990,000

Total poll of individuals: 9,000 + 1,000 + 99,000 + 891,000 = 1,000,000

We can now calculate the probability that if an individual tests positive, he has in fact the disease, that is, $P(D|T[positive]) = \frac{P(D \cap T[positive])}{P(T[positive])}$. Given the numbers above, the individuals who test positive are 9,000+99,000 and the individuals who test positive *and* have the disease are 9,000. So, $P(D|T[positive]) = \frac{P(D \cap T[positive])}{P(T[positive])} = \frac{9{,}000}{9{,}000+99{,}000} = 0.08333333333 \approx 8.3\%$.

We can also visualize the reasoning, as follows:



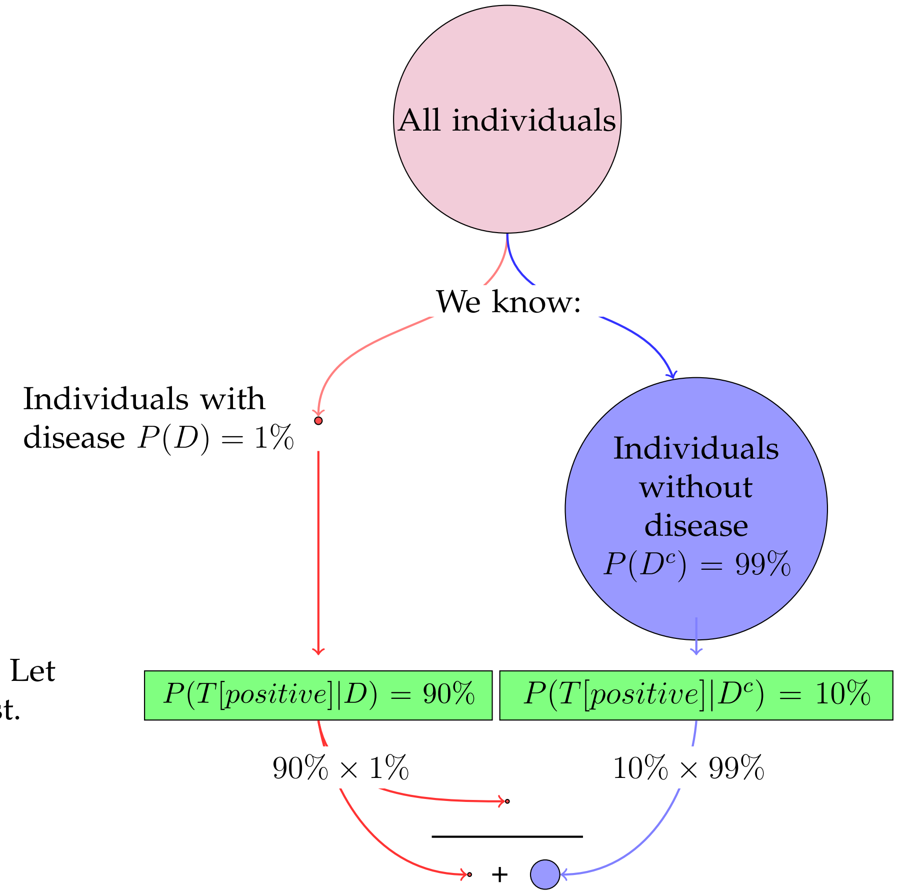
Now we test both groups. Let $T[positive]$ be a positive test.

Finally, to find the probability that a positive test *means having the disease*, we look at those who got a positive test *and have the disease*, and divide by all who got a positive test, disease or not.

$$\frac{9,000}{9,000 + 99,000} \approx 8.3\%$$

Surprisingly, the probability of having the disease given that the test is positive is still quite low. Instead, the probability of not having the disease given that the test is positive, is very high, roughly 91.7%. Recall that $P(D^c|T[positive]) = 1 - P(D|T[positive])$. If one tests positive, one is more likely *not* to have the disease than having it despite the test being 90% reliable!

II. Method of calculating probabilities (Bayes' theorem) We can also arrive at the same conclusion by calculating probabilities. We draw a similar diagram as before using probabilities:



Now we test both groups. Let $T[positive]$ be a positive test.

Finally, to find the probability that a positive test *means having the disease*, we look at those who got a positive test *and have the disease*, and divide by all who got a positive test, disease or not.

$$\frac{P(T[positive]|D])P(D)}{P(T[positive]|D)P(D) + P(T[positive]|D^c)P(D^c)} = \frac{90\% \times 1\%}{(90\% \times 1\%) + (10\% \times 99\%)} \approx 8.3\%$$

Bayes' theorem reflects the calculations above. This is Bayes' theorem applied to our case:

$$P(D|T[positive]) = \frac{P(T[positive]|D)P(D)}{P(T[positive]|D)P(D) + P(T[positive]|D^c)P(D^c)}.$$

The problem gave us all the probabilities to plug in, that is:

(i) the probability $P(D)$ regardless of $T[positive]$, that is, $P(D) = 1\%$

(ii) the probability of $T[positive]$ given D, i.e. $P(T[positive]|D) = 90\%$

(iii) $P(T[positive]|D)P(D) + P(T[positive]|D^c)P(D^c) = 90\% \times 1\% + 10\% \times 99\%$

By putting everything together, we get

$$P(D|T[positive]) = \frac{90\% \times 1\%}{(90\% \times 1\%) + (10\% \times 99\%)} \approx 8.3\%$$

D. EXERCISE: TAXI CABS

Imagine that there are two taxi companies, Green Cabs Inc. and Blue Cabs Inc., whose vehicles are respectively painted green and blue. Green Cabs Inc. covers 85 % of the market and Blue Cabs Inc. covers the rest. There are no other taxi companies around. On a misty day a cab hits and injures a passerby, but it drives off. A witness reports that it was a blue cab. The witness is right only 80 percent of the time. This means that his *reliability* equals 0.8 in the sense that he gets the color right 80 percent of the time. Given the witness report, what is the probability that the taxi cab involved in the accident was in fact blue?

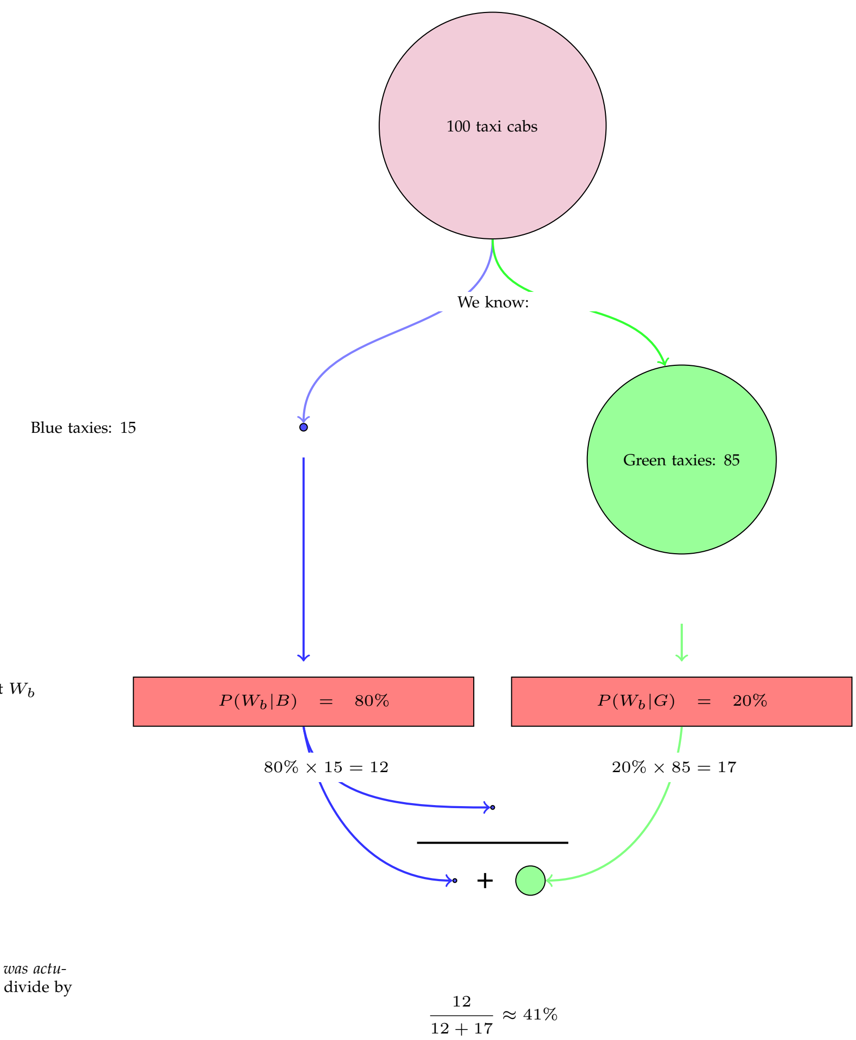
Demonstrate your conclusion by the method of counting cases but also by Bayes' theorem. Make sure you first correctly identify the probabilities of interest.

Solution to Taxi Cabs Here are some abbreviations:

B means "the taxi that hit the passerby belonged to Blue Cabs Inc."
G means "the taxi that hit the passerby belonged to Green Cabs Inc."
W_b means "the witness says the taxi that hit the passerby was blue"
W_g means "the witness says the taxi that hit the passerby was green"

We need to calculate $P(B|W_b)$. We are told that the witness is 80% reliable, that is, $P(W_b|B) = 80\%$ and $P(W_g|G) = 80\%$. So, $P(W_b|G) = 20\%$ and $P(W_g|B) = 20\%$. But, the problem does not give us information about how many taxies belonging to Green Cabs Inc. and Blue Cabs Inc. there are. We cannot calculate the required probability unless we have this information. The problem is not solvable.

We are told that Green Cabs Inc. covers 85 % of the market and Blue Cabs Inc. covers the rest. Given this assumption, $P(G) = 85\%$ and $P(B) = 15\%$. We can now solve the problem by counting cases. Consider 100 taxi cabs, as follows:



Now we suppose a witness sees a blue or green taxi. Let W_b be "witness says the taxi was blue"

Finally, to find the probability that W_b *means the the taxi was actually blue*, we look at those cases in which W_b *and* B, and divide by cases in which W_B, whether B or not.

We can arrive at the same result by Bayes' theorem, as follows:

$$P(B|W_b) = \frac{P(W_b|B)P(B)}{P(W_b)} = \frac{P(W_b|B)P(B)}{P(W_b|B)P(B)+P(W_b|G)P(G)} = \frac{80\%\times 15\%}{80\%\times 15\%+20\%\times 85\%} \approx 41\%.$$