Derivability

$\vdash \psi$

$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$

Intro to Logic - Midterm Review (2)

What We've Done So Far

Key notions

Syntax and **Semantics** of propositional logic

- Inductive definition of formulas
- $\mathbf{V} \models \psi$
- $\models \psi$
- $\phi_1, \phi_2, \ldots, \phi_k \models \psi$

Derivations in propositional logic

- Derivation rules
- $\vdash \psi$
- $\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$

Rules of Derivation in Propositional Logic

Reiteration

$$\frac{\phi}{\phi}\,R$$

If you have derived a formula, you can repeat it in the next line

$$\frac{\begin{array}{c}\phi\\ \vdots\\ \phi\end{array}}{\phi}\,R$$

WRONG use of R!

Notation

Greek letters such as ϕ, ψ or σ are used to represent **any formula**.

Now, if you see an expression such as $\phi \wedge \psi$ in the statement of a derivation rule, $\phi \wedge \psi$ represents any formula that has the shape of a conjunction such that the conjuncts ϕ and ψ can be any formula. The same goes for expressions such as $\phi \vee \psi$, $\phi \rightarrow \psi$, or $\neg\phi$

$\neg\phi$ is an abbreviation of $\phi \rightarrow \bot$

Rules for ∧

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\wedge\mathbf{I}$$

$$\frac{\phi \wedge \psi}{\phi}\wedge\mathbf{E}$$

$$\frac{\phi \wedge \psi}{\psi}\wedge\mathbf{E}$$

Derivation rules are introduced for the different connectives and there are **introduction rules** and **elimination rules** such as ∧**I** and ∧**E**

Rules for →

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow E$$

$$\frac{\begin{array}{c}[\phi]^i \\ \vdots \\ \psi\end{array}}{\phi \rightarrow \psi} \rightarrow I^i$$

This is *modus ponens*

This rules says that if you **assume** ϕ and then **manage to derive** ψ, you can **derive** $\phi \rightarrow \psi$ and **cancel assumption** ϕ

Derivations Rules Covered So Far

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\wedge I$$

$$\frac{\phi \wedge \psi}{\phi}\wedge E$$

$$\frac{\phi \wedge \psi}{\psi}\wedge E$$

$$\frac{\phi}{\phi}R$$

$$\frac{\begin{array}{c}[\phi]^i\\ \vdots\\ \psi\end{array}}{\phi \rightarrow \psi}\rightarrow I^i$$

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi}\rightarrow E$$

Derivability: $\vdash$

$\vdash \psi$ *iff*

there is a derivation of ψ in which all assumptions are canceled.

$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$ *iff*

there is a derivation of ψ from assumptions $\phi_1, \phi_2, \ldots, \phi_k$

Example (1)

$\vdash (\varphi \wedge \psi) \to (\psi \wedge \varphi)$

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$$\vdash (\varphi \wedge \psi) \to (\psi \wedge \varphi)$$

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\,\wedge E \quad \dfrac{[\varphi \wedge \psi]^1}{\varphi}\,\wedge E}{\psi \wedge \varphi}\,\wedge I}{(\varphi \wedge \psi) \to (\psi \wedge \varphi)}\,\to I^1$$

Example (2)

$\vdash \varphi \to (\psi \to \varphi)$

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$\vdash \varphi \to (\psi \to \varphi)$

$$\dfrac{\dfrac{\dfrac{[\psi]^1}{[\varphi]^2}}{\psi \to \varphi} \to I^1}{\varphi \to (\psi \to \varphi)} \to I^2$$

Example (3)

$$\vdash (\varphi \to \psi) \to ((\psi \to \sigma) \to (\varphi \to \sigma))$$

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$$\vdash (\varphi \to \psi) \to ((\psi \to \sigma) \to (\varphi \to \sigma))$$

$$\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi \to \psi]^3}{\psi} \to E \quad [\psi \to \sigma]^2}{\sigma} \to E}{\dfrac{\varphi \to \sigma}{(\psi \to \sigma) \to (\varphi \to \sigma)} \to I^1}{(\varphi \to \psi) \to ((\psi \to \sigma) \to (\varphi \to \sigma))} \to I^2 \to I^3$$

Example (4)

$$\vdash ((\varphi \wedge \psi) \to \sigma) \to (\varphi \to (\psi \to \sigma))$$

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$$\vdash ((\varphi \wedge \psi) \to \sigma) \to (\varphi \to (\psi \to \sigma))$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[(\varphi \wedge \psi) \to \sigma]^3 \quad \dfrac{[\varphi]^2 \quad [\psi]^1}{\varphi \wedge \psi}\wedge I}{\sigma}\to E}{\psi \to \sigma}\to I^1}{\varphi \to (\psi \to \sigma)}\to I^2}{((\varphi \wedge \psi) \to \sigma) \to (\varphi \to (\psi \to \sigma))}\to I^3$$