

Derivability

$\vdash \psi$

$\phi_1, \phi_2, \dots, \phi_k \vdash \psi$

Intro to Logic - Midterm Review (2)

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# What We've Done So Far

## Key notions

Syntax and  
Semantics of  
propositional logic

- ❖ Inductive definition of formulas
- ❖  $V \models \psi$
- ❖  $\models \psi$
- ❖  $\phi_1, \phi_2, \dots, \phi_k \models \psi$

Derivations in  
propositional logic

- ❖ Derivation rules
- ❖  $\vdash \psi$
- ❖  $\phi_1, \phi_2, \dots, \phi_k \vdash \psi$



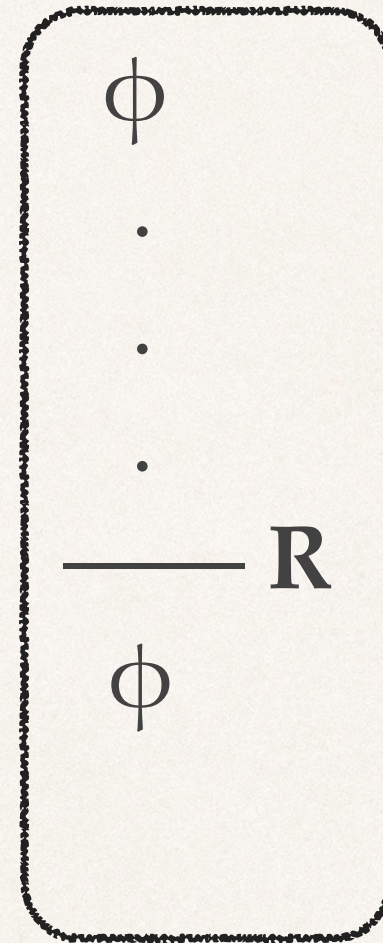
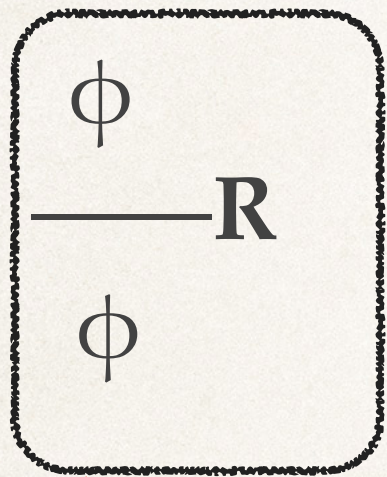
# Rules of Derivation in Propositional Logic

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# Reiteration

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**WRONG use of R!**

If you have derived a formula, you can repeat it in the next line



# Notation

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Greek letters such as  $\phi$ ,  $\psi$  or  $\sigma$  are used to represent **any formula**.

Now, if you see an expression such as  $\phi \wedge \psi$  in the statement of a derivation rule,  $\phi \wedge \psi$  represents any formula that has the shape of a conjunction such that the conjuncts  $\phi$  and  $\psi$  can be any formula. The same goes for expressions such as  $\phi \vee \psi$ ,  $\phi \rightarrow \psi$ , or  $\neg\phi$

$\neg\phi$  is an abbreviation of  $\phi \rightarrow \perp$



# Rules for $\wedge$

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$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge I$$

$$\frac{\phi \wedge \psi}{\phi} \wedge E$$

$$\frac{\phi \wedge \psi}{\psi} \wedge E$$

Derivation rules are introduced for the different connectives and there are **introduction rules** and **elimination rules** such as  $\wedge I$  and  $\wedge E$



# Rules for $\rightarrow$

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow \mathbf{E}$$

$$\frac{\begin{array}{c} [\phi]^i \\ \cdot \\ \cdot \\ \cdot \\ \psi \end{array}}{\phi \rightarrow \psi} \rightarrow \mathbf{I}^i$$

This is *modus ponens*

This rule says that if you **assume**  $\phi$  and then **manage to derive**  $\psi$ , you can **derive**  $\phi \rightarrow \psi$  and **cancel assumption**  $\phi$



# Derivations Rules Covered So Far

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$$\frac{\phi}{\phi} \text{R}$$

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge \text{I}$$

$$\frac{\phi \wedge \psi}{\phi} \wedge \text{E}$$

$$\frac{\phi \wedge \psi}{\psi} \wedge \text{E}$$

$$\frac{\begin{array}{c} [\phi]^i \\ \cdot \\ \cdot \\ \cdot \\ \psi \end{array}}{\phi \rightarrow \psi} \rightarrow \text{I}^i$$

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow \text{E}$$



# Derivability: $\vdash$

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$\vdash \psi$                       *iff*

there is a derivation of  $\psi$  in which all assumptions are canceled.

$\phi_1, \phi_2, \dots, \phi_k \vdash \psi$                       *iff*

there is a derivation of  $\psi$  from assumptions  $\phi_1, \phi_2, \dots, \phi_k$



# Example (1)

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$$\vdash (\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$$



# Example (1)

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$\vdash (\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$

$$\frac{\frac{\frac{[\varphi \wedge \psi]^1}{\psi} \wedge E}{\psi \wedge \varphi} \wedge I}{(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)} \rightarrow I^1$$



# Example (2)

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$$\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$$



# Example (2)

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$\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$

$$\frac{\frac{\frac{[\psi]^1}{[\varphi]^2}}{\psi \rightarrow \varphi} \rightarrow I^1}{\varphi \rightarrow (\psi \rightarrow \varphi)} \rightarrow I^2$$



# Example (3)

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$$\vdash (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\varphi \rightarrow \sigma))$$



# Example (3)

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$\vdash (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\varphi \rightarrow \sigma))$

$$\frac{\frac{\frac{[\varphi]^{1} \quad [\varphi \rightarrow \psi]^{3}}{\psi} \rightarrow E}{\frac{\sigma}{\varphi \rightarrow \sigma} \rightarrow I^{1}} \rightarrow I^{2}}{(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\varphi \rightarrow \sigma))} \rightarrow I^{3}}$$



# Example (4)

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$$\vdash ((\varphi \wedge \psi) \rightarrow \sigma) \rightarrow (\varphi \rightarrow (\psi \rightarrow \sigma))$$



# Example (4)

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$\vdash ((\varphi \wedge \psi) \rightarrow \sigma) \rightarrow (\varphi \rightarrow (\psi \rightarrow \sigma))$

$$\frac{\frac{\frac{[(\varphi \wedge \psi) \rightarrow \sigma]^3 \quad \frac{[\varphi]^2 \quad [\psi]^1}{\varphi \wedge \psi} \wedge I}{\sigma} \rightarrow E}{\psi \rightarrow \sigma} \rightarrow I^1}{\varphi \rightarrow (\psi \rightarrow \sigma)} \rightarrow I^2}{((\varphi \wedge \psi) \rightarrow \sigma) \rightarrow (\varphi \rightarrow (\psi \rightarrow \sigma))} \rightarrow I^3$$