PHIL 50 **A Step-by-step Guide to Two derivations**

Hope you all are doing well! We thought it might be good to provide you with some more examples of derivations since they will be on the midterm. To that end, we am providing this walkthrough of two derivations discussed in Adam's section last week to the entire class. This is just one of the few extra items that you will be given to help you study for your midterms. Please note that you ought to use your time wisely: derivations won't be the only thing on the test. (Perhaps we'll send a second handout on semantic issues surrounding propositional logic) Nevertheless since doing them can be one of more daunting aspects of the class, a walkthrough (as opposed to a set of completed derivations) might prove helpful. We certainly hope that this is so.

Before we get started, here are a couple notes:

(1) If you attended Adam's section last week, you've already received a previous version of the material here. There are only minor edits herein, so there's little need to restudy these derivations.
(2) You'll notice that there are some symbols that you've not previously encountered. This was for formatting reasons. Remember: **~ =** ¬ (negation); ^ = ∧ (conjunction); → = → (conditional/material implication); ←→ =↔ (bi-conditional/equivalence).
(3) You might think about constructing the derivations in a different way and that's fine!—we just know that these tips help us, and hopefully they'll help you as well when you get stuck.
(4) Regardless of whether you use this walkthrough for your midterm, it may be of some use in the future as you will be doing derivations in predicate logic.

Here we'll first provide a step-by-step walkthrough of the derivation of the following formula:

1. ((p^q)→(~((~p) v (~q))))

We'll also provide (in two separate attachments) the initial sloppy sketch of the derivation, and then how it should look in its final form. Afterward, we'll tackle the derivation of the following formula from your last homework:

2. ((p→q)←→p)→q

You can look to Marcello's solutions to last week's homework for the full graphical presentation of the derivation (it's problem 2b of Homework 3). Still, everything is here that you need to construct the entire proof yourself. Lastly, we'll just give some advice of constructing derivations in this tree-like natural deduction form.

You are asked to derive:

((p^q)→(~((~p) v (~q))))

The first thing we do is copy the formula I am supposed to prove on the very bottom of the derivation we want to construct. So give yourself some space and at the bottom of this space, copy the formula. Then, since you need to derive it, a line should go above the formula (remember it is the conclusion of an argument). At the bottom then, you ought to have the following:

$$\frac{}{((p\hat{}q)\rightarrow(\sim((\sim p)\ v\ (\sim q))))}$$

Now we think about how we might end up with this conclusion. What we look for, personally, is the main connective of what we wish to prove. So what's the main connective of ((p^q)→(~((~p) v (~q))))?—It's →. Then we think, well, what derivation rules include a →? Only →-into (i.e., → I) and →-elim (i.e., → E), obviously. Which of these should we use for our proof? Well it seems that we should use intro since our conclusion contains a →. If that's the case and since the → I looks like this:

$$\begin{array}{c} [\varphi] \\ \vdots \\ \Psi \\ \hline \Phi\rightarrow\psi \quad \rightarrow I \end{array}$$

We need to write the consequent of ((p^q)→(~((~p) v (~q)))) on the line directly above the final conclusion and put an assumption of the antecedent somewhere above (leave room for yourself since you can always figure out how to draw a pretty tree derivation after you have all the steps to the proof in place). But the question now becomes: which the consequent and which is the antecedent of ((p^q)→(~((~p) v (~q))))?—It should be clear by now that the antecedent is (p^q) while the consequent is (~((~p) v (~q))). So (per the explanation of → I above) write the latter immediately above the last line and put the former as an assumption somewhere above that. Number the assumption with superscript 1, and write →I on the side of the line, also with superscript 1. Your derivation should now look like this:

$$\begin{array}{c} [p\hat{}q]^1 \\ \vdots \\ (\sim((\sim p)\ v\ (\sim q))) \qquad \rightarrow I^1 \\ \hline ((p\hat{}q)\rightarrow(\sim((\sim p)\ v\ (\sim q)))) \end{array}$$

So now focus on what's above the line, namely (~((~p) v (~q))). You have to derive this statement, somehow, from you assumption (p^q). How?—Well, look at the main connective for this new formula ((~((~p) v (~q))); what is it? It's obviously a negation, since that's what's out in front. But how do you introduce a negation? Well remember that, in general, ~φ = φ→ ⊥, so it looks like we need another →I rule. To make sure that you're not confused let's write this all out noting that (~((~p) v (~q))) = ((~p) v (~q))→ ⊥.

[p^q]^{1}

¦

((~p) v(~q))→⊥ →I^{1}

((p^q)→(~((~p) v (~q))))

Now that we've changed that formula [(~((~p) v (~q)))] to something equivalent [((~p) v(~q))→⊥], we can try moving on with the proof. Remember that now we are trying to derive ((~p) v(~q))→⊥. To derive this, you first write another line above it. This will give you the following [Note: to focus attention on this step of the proof we may omit what comes below. You'll see what the whole thing looks like in the separate attachment]:

[p^q]^{1}

¦

((~p) v(~q)→⊥

So what's the main connective in ((~p) v(~q))→⊥? Obviously, →. So what rule do we use?—Well, it looks like →I again. So you identify the antecedent and consequent of the formula and write the consequent immediately above the line and the antecedent as another assumption farther above. Be sure to include the correct superscripts on this new assumption and the rule to the side of this step. This should give you the following:

[p^q]^{1}

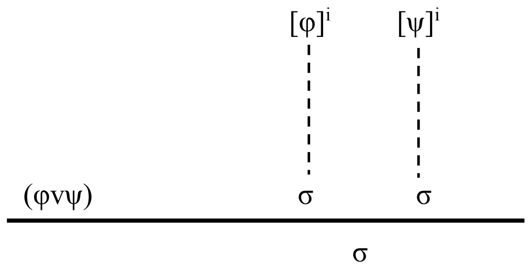
[((~p) v(~q))]^{2}

¦

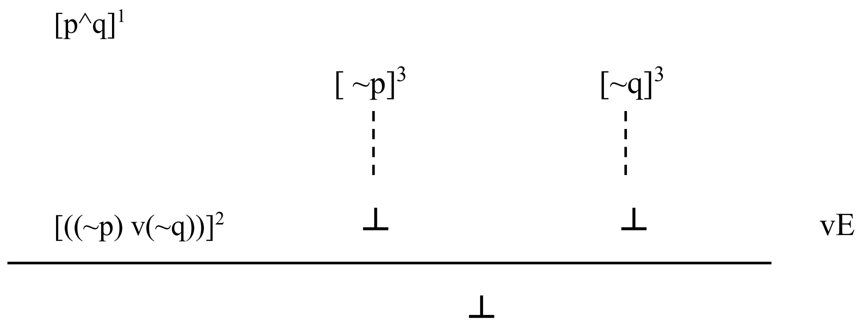
⊥ →I^{2}

((~p) v(~q))→⊥

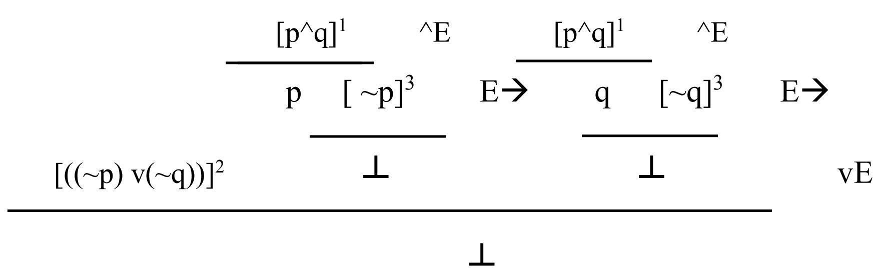
Now you have to find out how to derive a contradiction from your second assumption, ((~p) v(~q)). How do you do this? Well, if you look at the main connective for this formula, you'll notice it's a disjunction (v). You'll have to get rid of it some way. How? Well, disjunction elimination (vE) suggests itself. The form of the rule looks like this:



Remember that, with this rule, we assume each disjunct, bracket it but use the same superscript; technically, this is unnecessary but it is standard practice for this class. The question now is what goes into the sigma (σ) spot for the purposes of our proof. Well, since we wish to derive a contradiction from the disjunction ((~p) v(~q)), it seems that we want ⊥ to fill that spot. So, in our proof of ⊥ from ((~p) v(~q)), we have the following line (we likewise leave off whatever occurs below this step—proof of ⊥-- since well put it all together later).



There are two final questions: (a) how to derive ⊥ from ~p, and (b) how to derive ⊥ from ~q? The key is to look at your first assumption [p^q][1] . Notice that conjunction elimination (^E) allows you to derive both p and q by themselves. This is exactly what you need to prove a contradiction in your two disjunction elimination subproofs. So, how does this go exactly? Like this:



Putting these pieces together, you should now have a complete derivation of the original formula. Be sure and check to make sure that all of your assumptions are closed. If not, you might not have canceled something, or you did something else wrong (e.g., introduced an unnecessary assumption). The full picture of the proof will be provided in a picture (since it sucks trying to format these things on a word processor).

1. You are asked to prove the following formula:

   $((p\rightarrow q)\leftarrow\rightarrow p)\rightarrow q$

   You know that this is going to be at the very bottom of your proof so you should leave yourself some room and copy it with a line over it (since you'll be deducing it from an inference step). Same sort of thing as last time; you should have this:

$$\frac{\qquad\qquad}{((p\rightarrow q)\leftarrow\rightarrow p)\rightarrow q}$$

   Now, just like last time, you should identify the main connective. Here it's the last →. So, which inference rule should we use? The most natural are those involving →, but which one? Since → occurs in the bottom as the main connective, it makes the most sense to think that the step above *introduced* that connective to construct the final formula. So we go with → I. What next?—Remember that you should locate the consequent for this rule and put it directly above the line, and then locate the antecedent and put it somewhere above as a numbered assumption. Then to the side of the line you copy the abbreviation of the rule along a superscript corresponding to the number of the assumption. It should now look like this:

$$\begin{array}{c} [((p\rightarrow q)\leftarrow\rightarrow p)]^{1} \\ \vdots \\ \dfrac{q}{((p\rightarrow q)\leftarrow\rightarrow p)\rightarrow q} \end{array} \qquad \rightarrow I^{2}$$

   What do you do next? Well, your formula q, which you now have to derive, doesn't include any main connectives since it's atomic. But, given the explanation in the problem, there is something you can already do, and that's unpack your bi-conditional assumption so it contains only a conjunction of two conditional statements. Just focusing on the assumption, it should look like this:

$$\frac{[((p\rightarrow q)\leftarrow\rightarrow p)]^{1}}{((p\rightarrow q)\rightarrow p)\wedge(p\rightarrow(p\rightarrow q))}\ \text{unpack} \leftarrow\rightarrow$$

Now do you see anything from which you might be able to derive q? Well, there's one thing that might help—the (p→q) that occurs in this formula twice. If you could somehow perform a →E on that conditional to get the q then you would have done everything you needed for the proof. Notice that to utilize this rule you also need the antecedent of the condition (p) besides the conditional p→q. Let's try this. We now get the following:

$$\dfrac{[((p\rightarrow q)\leftrightarrow p)]^{1}}{((p\rightarrow q)\rightarrow p)\wedge(p\rightarrow(p\rightarrow q))}\ \text{unpack}\leftrightarrow$$

$$\vdots$$

$$\dfrac{\dfrac{p \qquad (p\rightarrow q)}{q}\ \rightarrow E}{((p\rightarrow q)\leftrightarrow p)\rightarrow q}\ \rightarrow I^{2}$$

But now to complete the proof we need to do two things derive (1) derive p from assumption 1, and (2) derive (p→q) from assumption 1. Well, it seems that now our derivation its going to branch and become tree-like where one derivation will prove p and the other will prove (p→q). Be aware of this when structuring your proof. Let's handle each tree separately, and you can combine it later. To that end, let's prove p from our open assumption (remember it doesn't get closed until the bottom introduction step and so you can still utilize it in your proof). Let's see—p, like the q below it, does not have a main connective. So how might we get it? Well, look back at the unpacked open assumption for a hint; it seems that there is something that we can derive p from—the first conjunct—and it also seems like the way to do that is by way of →E. Let's try it. When we do we get the following:

$$\dfrac{(p\rightarrow q) \qquad (p\rightarrow q)\rightarrow p}{p}\ \rightarrow E$$

Well, that's all well and good, but now it appears that we have to proof both p→q and (p→q)→q from our open assumption. And again, these 'sub-derivations' are going to branch off in their own directions. Let's take p→q first.

Ask yourself: does p→q have a main connective? Obviously, it does and obviously it's →, so you know the drill. The rule is probably going to be →I, in which case you write the whole formula then a line above it, then the consequent immediately above that, assume the antecedent somewhere (leave yourself space) above that with another superscript, then cite the rule →I with a matching superscript next to the line. It should look like this:

[p]²

¦

q →I²

―――――――――

(p→q)

Ok, so we have to prove q again (like you have to do below). How do we do it here? Well, look at your open assumptions. They are the unpacked ((p→q)→p)^ (p→(p→q)) and p. Well, again it seems nice if we could prove q by way of a p→q, which is to say, by way of conditional elimination. To do it, you know you also need the antecedent p, but luckily you have that as an open assumption above, so all you need to do is provide a derivation of p→q. Let's try it:

[p]²

p (p→q)

―――――――――

q →I²

―――――――――

(p→q)

Now we have one last p→q that you have to prove at this level of subproof (but remember you have to also do it below in a different tree!). So how do we do it? Look again at our two open assumptions ((p→q)→p)^ (p→(p→q)) and p. It seems that we can get p→q by way of the second conjunct and p—all we need is a conditional elimination step. So we have

[p]² [((p→q)←→p)]¹ unpack ←→

―――――――――

((p→q)→p)^ (p→(p→q)) ^E

―――――――――

(Remember you can repeat unclosed assumptions) [p]² p→(p→q) →E

―――――――――

[p]² (p→q) →E

―――――――――

q

―――――――――

(p→q) →I²

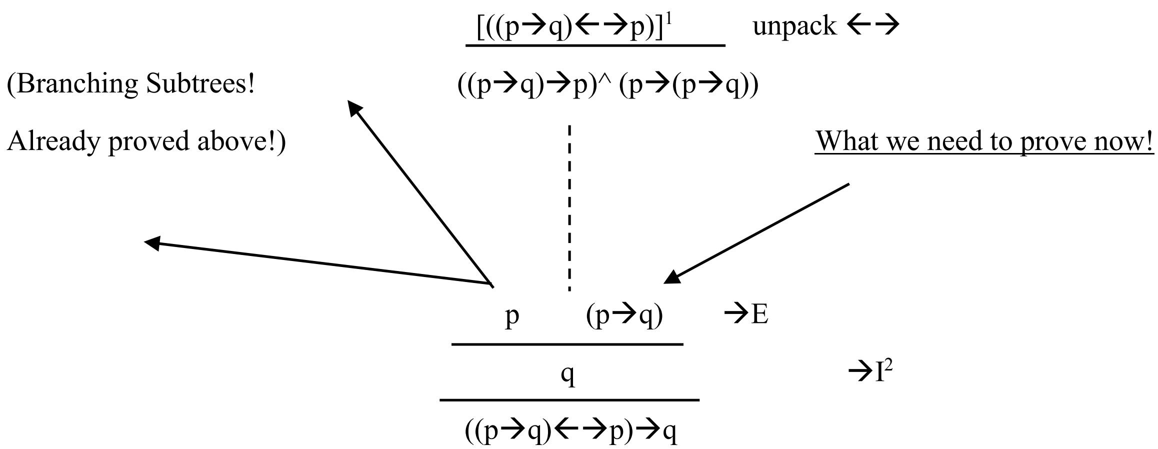
So now you've completed one branch (of a branch!) of your derivation. You have one more branch of a branch to prove until you're done with this entire side of the derivation. Go back to (p→q)→p. Remember that it's something that we have to prove in order to use →E and get p, and that we've just proved it's antecedent (p→q). Luckily, if we look at our open assumptions, the small branch is easy since the needed formula occurs as a conjunct in our unpacked bi-conditional assumption. So we just do the following:

[((p→q)← →p)]^1 unpack ←→

((p→q)→p)^ (p→(p→q))   ^E

(p→q)→p

So, now we're back to the other main branch wherein we have to prove (p→q). We'll recopy it to rekindle your memory:



Well, we want to prove (p→q) so how do we do that?—That's right, think of conditional introduction (since the main connective is →). Write what you're trying to prove (i.e., p→q)—which you probably have down already—and put a line above. Then put the consequent immediately above your drawn line, and the antecedent as a numbered assumption somewhere above that (leave yourself room!). Finally, cite the →I step with a matching numbered superscript next to your drawn line. That gives you this:

[p]^3

⋮

q   →I^3

(p→q)

Note that even though this looks like a skeleton of a subproof above (for a different branch) with the same proposition assumed (i.e., p), it is nevertheless a different assumption! This is because that assumption is open and closed within that branch of the proof and is therefore not available to use in this branch. So you have to make a new assumption with a new number, but of the same proposition. In any case, now we again have q. By now you should know how to go about proving it from your two open assumptions [p]^{3} and [((p→q)→p)^ (p→(p→q))]^{1}. You first use conjunction elimination on the unpacked assumption getting (p→(p→q)), then you use assumption p and this (p→(p→q)) to get (p→q) by way of conditional elimination. Then you use assumption p again and (p→q) to get q by way of conditional elimination. You see this all below.

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        [((p→q)←→p)]^1              unpack ←→
  ─────────────────────────────
    ((p→q)→p)^ (p→(p→q))            ^E
  ─────────────────────────────
      [p]^3    (p→(p→q))               →E
  ─────────────────────────────
    [p]^3       (p→q))              →E
  ─────────────────────────────
               q
  ──────────────────────
            (p→q)            →I^3
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You now have all the steps to the proof for this problem. It will be up to you to put the steps together in a manner consistent with the geometry of tree construction so that you have a complete derivation.

General Advice for derivations:

--Work backwards, from bottom to top.

--Focus on individual branches of derivations when you get to them as if they were independent derivations that you have to do.

--Look at the formula you have to prove at the step you have to prove it.—If it has a main connective then you should probably use a rule corresponding to that main connective's intro rule. If this doesn't work, then look at your assumptions and see if you can derive the formula you want from the assumptions. As a last resort, assume the formula's negation and prove that it leads to a contradiction given your open assumptions (this might require completing further sub-proofs).

--Make sure that all of your assumptions, when you think you're done with the proof, are closed. If you've worked backwards, this should be pretty easy to do since you will have introduced assumptions (as you worked upwards) only by citing a rule which closes it.

--Memorize the introduction and elimination rules for each connective!—You might not be able to complete the derivation, or might not be able to complete it correctly, without this knowledge. Remember that, though there are rationales behind having the set of rules that we currently have rather than others, you should treat the inference rules like rules of a game. When playing monopoly, chess, or any other game we usually don't inquire into the rationale for any particular rule. Rather, these rules are constitutive of the particular game you're playing, i.e., if you had different rules, then you'd be playing a different game. So just treat these rules of inference that you can use in derivations like rules of a game. Everything becomes more mechanical the more you play.