PHIL 50 – INTRODUCTION TO LOGIC

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DERIVATIONS IN PREDICATE LOGIC – WEEK #8

1 FREE AND BOUND VARIABLES

Before discussing the derivation rules for predicate logic, we should distinguish between *free variables* and *bound variables*. Intuitively, a variable x occurs free in a formula if there is no quantifier that binds x. For example, the variable x occurs free in the formula $P(x)$ because there is no quantifier that binds x. In the formula $\forall x P(x)$, instead, the variable x is bound by the universal quantifier $\forall x$. More precisely:

> A variable x occurs free provided x is not within the scope of the quantifier $\forall x$ or $\exists x$. By contrast, a variable x is bound if it does occur within the scope of the quantifier $\forall x$ or $\exists x$.

What is the scope of a quantifier? To determine the scope of a quantifier, look at the open bracket '(' that immediately follows the quantifier and wait until the bracket is closed by ')'. The formula within the brackets (...) is the scope of the quantifier. For example, the scope of the quantifier $\exists x$ in the formula $\exists x((R(x, y) \wedge P(x) \wedge A(y))$ is the formula $R(x, y) \wedge P(x) \wedge A(y)$.

Here are some examples of the distinction between free and bound variables:

- The variable x occurs free in $\forall y(P(x))$ because the universal quantifier $\forall y$ does not bind the variable x. Importantly, the variable x is within the scope of the quantifier $\forall y$ because x is part of $P(x)$, but x cannot be bound by a quantifier such as $\forall y$. By contrast, the variable y is bound by the quantifier $\forall y$ in the formula $\forall y(P(y))$.
- Consider the formula $\exists x(P(x) \wedge P(y))$. The variable x is bound by the quantifier $\exists x$, although the variable y occurs free because there is no quantifier that binds y. Instead, in the formula $\exists y(\exists x(P(x) \wedge P(y)))$, both variables x and y are bound by a quantifier.[1]
- Consider the formula $\exists x(P(x)) \wedge \exists x \neg(P(x))$. The first occurrence of x is bound by the first existential quantifier, while the second occurrence of x is bound by the second existential quantifier. The formula $\exists x(P(x)) \wedge \exists x \neg(P(x))$ is no different from

[1] Note that, relative to the formula $\exists y(\exists x(P(x) \wedge P(y)))$, the scope of $\exists y$ is the formula $\exists x(P(x) \wedge P(y))$, while the scope of $\exists x$ is the formula $P(x) \wedge P(y)$. In this case, we say that the quantifiers are nested because one occurs within the scope of the other.

$\exists x(P(x)) \wedge \exists y \neg (P(y))$. In the latter formula, two variables x and y are used, whereas in the former formula, two occurrences of the same variables x are used.

- Consider the formula $\exists x(P(x)) \wedge \exists y \neg (R(x, y))$. Is the second occurrence of x bound or free? The only quantifier that could bind the second occurrence of x is $\exists x$. However, if you look at the brackets carefully, you'll see that the scope of $\exists x$ is $P(x)$, so the second occurrence of x is outside the scope of $\exists x$. Hence, the second occurrence of x occurs free.

2 NOTATIONAL CONVENTIONS

We shall follow some notational conventions. We shall denote formulas of predicate logic of arbitrary complexity by means of Greek letters such as φ, ψ, σ, etc. (If you are curious, here is the Greek alphabet: $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu, \nu, \xi, o, \pi, \rho, \sigma, \tau, \upsilon, \varphi, \chi, \psi, \omega$.)

We shall write $\varphi(x)$, $\psi(x)$, $\sigma(x)$, etc. to denote a formula of arbitrary complexity in which the variable x occurs free. For example, consider the formula $\forall x(P(x) \wedge R(y, x))$. This is a formula in which x is bound but y occurs free, so we can refer to the formula through $\varphi(y)$.

Another notational convention is that if we write $\forall x\varphi(x)$, $\forall x\psi(x)$, $\forall x\sigma(x)$, etc. we mean that the variable x in $\varphi(x)$, $\psi(x)$, $\sigma(x)$, etc. is universally quantified. And if we write $\exists x\varphi(x)$, $\exists x\psi(x)$, $\exists x\sigma(x)$, etc. we mean that the variable x in $\varphi(x)$, $\psi(x)$, $\sigma(x)$, etc. is existentially quantified.

A final notational convention is that we shall use the expression 't' to denote a term. A term can be either a variable symbol x, y, z or a constant symbol a, b, c. So, the expression 't' is a generic placeholder for a variable symbol or a constant symbol.

3 DERIVATION RULES FROM PROPOSITIONAL LOGIC

The rules of derivation for predicate logic include the derivation rules for propositional logic, repeated here for your convenience:

REITERATION

(R)

$$\frac{\varphi}{\varphi}\ R$$

RULES FOR $\wedge$

($\wedge I$)

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ \wedge I$$

($\wedge E$)

$$\frac{\varphi \wedge \psi}{\varphi}\ \wedge E$$

$$\frac{\varphi \wedge \psi}{\psi}\ \wedge E$$

RULES FOR $\rightarrow$

($\rightarrow I$)

$$\frac{\begin{array}{c}[\varphi]^i \\ \vdots \\ \psi\end{array}}{\varphi \rightarrow \psi}\rightarrow I^i$$

($\rightarrow E$)

$$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi}\rightarrow E$$

RULES FOR $\perp$

($\perp$)

$$\frac{\perp}{\psi}\ \perp$$

(RAA)

$$\frac{\begin{array}{c}[\neg\varphi]^i \\ \vdots \\ \perp\end{array}}{\varphi}\ RAA^i$$

RULES FOR $\vee$

$(\vee I)$

$$\frac{\varphi}{\varphi \vee \psi}\ \vee I$$

$$\frac{\psi}{\varphi \vee \psi}\ \vee I$$

$(\vee E)$

$$\dfrac{\varphi \vee \psi \quad \begin{matrix}[\varphi]^i \\ \vdots \\ \sigma\end{matrix} \quad \begin{matrix}[\psi]^i \\ \vdots \\ \sigma\end{matrix}}{\sigma}\ \vee E^i$$

Concerning the application of the above rules, almost nothing changes from propositional logic. The only difference is that the Greek letters such as φ, ψ, σ, etc. should now be understood as placeholders for formulas of predicate logic (and not for formulas of propositional logic). To the rules from propositional logic, four more derivation rules are added which are specific to predicate logic.

4 DERIVATIONS RULES FOR $\forall$

The first two additional rules are for the elimination and for the introduction of the universal quantifier, as follows:

$(\forall E)$

$$\frac{\forall x \varphi(x)}{\varphi(t)}\ \forall E$$

$(\forall I)$

$$\frac{\varphi(x)}{\forall x \varphi(x)}\ \forall I$$

> **RESTRICTION ON** $\forall I$. Variables x does not occur free in any uncanceled assumption on which $\varphi(x)$ depends.

Let's consider each rule in turn. Rule $\forall E$ says that if you have a derivation of a universally quantified formula $\forall x \varphi(x)$, then you also have a derivation of a formula without the universal quantifier where variable x is replaced by some term t. Recall that t can be a variable but it can also be a constant symbol. The intuitive meaning of $\forall E$ is this. If we can claim that everybody is φ, then we can also claim that someone in particular is φ. Let's look at an example. Suppose we can claim that everybody from Alaska is an expert skier. This statement can be translated in predicate logic as $\forall x((A(x) \to S(x))$, where A is the predicate for being from Alaska and S is the predicate for being an expert skier. Let's consider a particular guy named Tyler, using the constant symbol $tyler$. Now, an application of rule $\forall E$ to our case looks as follows:

$$\dfrac{\forall x(A(x) \to S(x))}{A(tyler) \to S(tyler)}\ \forall E$$

All we have done is eliminate the universal quantifier and replace variable x with the constant symbol $tyler$. If we add the additional premise that Tyler is indeed from Alaska, by the rule $\to E$, we can derive the conclusion that Tyler is an expert skier, as follows:

$$\dfrac{A(tyler) \qquad \dfrac{\forall x(A(x) \to S(x))}{A(tyler) \to S(tyler)}\ \forall E}{S(tyler)}\ \to E$$

The above derivation rests on two uncanceled assumptions, i.e. $\forall x(A(x) \to S(x))$ and $A(tyler)$.

We now turn to the second rule for the universal quantifier, namely $\forall I$. This is a delicate rule and it is important not to misunderstand it. First of all, the rule cannot always be applied, but it only applies when a particular restriction is satisfied. The restriction is repeated here for convenience:

> **RESTRICTION ON** $\forall I$. Variables x does not occur free in any uncanceled assumptions on which $\varphi(x)$ depends.

To see the importance of the restriction, let's consider a slightly modified version of the above derivation, as follows:

$$\dfrac{\dfrac{A(x) \qquad \dfrac{\forall x(A(x) \to S(x))}{A(x) \to S(x)}\ \forall E}{S(x)}\ \to E}{\forall x S(x)}\ \textbf{WRONG!}$$

The last step in the derivation is wrong. Why? Well, it is an attempt to use the rule $\forall I$ while the restriction is not satisfied. Note that $S(x)$ depends on the uncanceled assumption $A(x)$, which contains a free occurrence of x. This is a violation of the restriction. In contrast, consider another derivation, as follows:

$$\dfrac{\dfrac{\dfrac{\forall x A(x)}{A(x)}\ \forall E \qquad \dfrac{\forall x(A(x) \to S(x))}{A(x) \to S(x)}\ \forall E}{S(x)}\ \to E}{\forall x S(x)}\ \forall I$$

What has changed here? We still have an uncanceled assumption, namely $\forall x A(x)$, together with the other uncanceled assumption $\forall x(A(x) \rightarrow S(x))$. The difference here is that the uncanceled assumptions on which $S(x)$ depends do not contain any free occurrence of x. In both $\forall x A(x)$ and $\forall x(A(x) \rightarrow S(x))$, the variables x is bound and not free. And since the variable x does not occur free any longer, the restriction for the application of $\forall I$ is now satisfied.

The difference between the two derivations—the difference between the incorrect and the correct derivation—should be intuitive. In the wrong derivation, we tried to conclude that everybody is an expert skier from the assumption that some x is an Alaskan and that all Alaskans are expert skiers. But that's clearly wrong. In the correct derivation, we concluded that everybody is an expert skier from the assumption that everybody is Alaskan, i.e. $\forall x A(x)$, and the assumption that all Alaskans are expert skiers. This is a good inference. (Of course, we can challenge the assumption that everybody is Alaskan, but that's another matter. If everybody is in fact Alaskan, and if every Alaskan is in fact an expert skier, it undeniably follows that everybody is an expert skier.)

The derivation rule $\forall I$—together with the relevant restriction—codifies an important feature of mathematical reasoning and of formal reasoning more generally. It is the idea that if we can show that a certain property holds for an arbitrary object of a certain type, then that property holds for all objects of that type. For example, if we can show that *for an arbitrary triangle* the sum of its internal angles is equal to two right angles, then it follows that *for all triangles* the sums of the internal angles is equal to right angles. (This is what Euclid does in proposition I.32 of the *Elements*.) In other words, rule $\forall I$ codifies the reasoning step that moves **from** "*for an arbitrary* x , $\varphi(x)$" **toward** "*for all* x , $\varphi(x)$".

But what is an arbitrary object of a certain type? The idea of an arbitrary object that is referred to by x is captured in the rule $\forall I$ by the relevant restriction. The restriction says that x cannot occur free in any uncanceled assumptions on which $\varphi(x)$ depends. If x were to occur free in some uncanceled assumption on $\varphi(x)$ depends, that would mean that certain additional assumptions have been made about x, thereby making x not arbitrary any longer. Now, x cannot refers to a completely arbitrary object; x refers to an arbitrary of a certain type. After all, we are considering the formula $\varphi(x)$, and so we are putting certain restrictions on x, e.g. x being a triangle, or a prime number, or an Alaskan, or whatever.

5 RULES FOR $\exists$

($\exists I$)

$$\frac{\varphi(t)}{\exists x \varphi(x)}\ \exists I$$

($\exists E^i$)

$$\dfrac{\exists x\varphi(x) \qquad \begin{array}{c}[\varphi(x)]^i\\ \vdots \\ \psi\end{array}}{\psi}\ \exists E^i$$

RESTRICTION ON $\exists I$. Variables x does not occur free in ψ and x does not occur free in any uncanceled assumption in the sub-derivation of ψ except for $\varphi(x)$.

Let's consider each rule in turn. Rule $\exists I$ is straightforward. The rules says that if you have a derivation of $\varphi(t)$ with t a variable or a constant symbol, then you also have a derivation of $\exists x\varphi(x)$. This is hardly problematic. For instance, if you have a derivation that Tyler is an expert skier, i.e. $S(tyler)$, you can also conclude that there is someone who is an expert skier, i.e. $\exists x S(x)$. That's all rule $\exists I$ is saying.

Understanding rule $\exists E$ is instead more difficult and subtle. In fact, some introductory logic courses omit $\exists E$ because it is difficult and hard to grasp. But that's not going to happen in this course! To begin with, it is important to see what rule $\exists E$ allows us to do. Suppose you have a derivation of the existentially quantified formula that there are some expert skiers, i.e. $\exists x S(x)$. What can you derive from $\exists x S(x)$? Well, in accordance with rule $\exists E$, if you assume $S(x)$ and manage to derive a formula ψ, then you can derive ψ from $\exists x S(x)$ while canceling the assumption that $S(x)$ (*provided some restrictions are met!*). So, for instance, suppose you assume that x is a skier, i.e. $S(x)$. And suppose you have available the additional assumption that every skier is a human being, i.e. $\forall x(S(x) \to H(x))$. Through some reasoning steps, you can arrive at the conclusion that there is a human being, i.e. $\exists x H(x)$. Finally, through rule $\exists E$, you can derive $\exists x H(x)$ and cancel the assumption that $S(x)$. This is what the reasoning looks like in the form of a derivation:

$$\dfrac{\exists x S(x) \qquad \dfrac{\dfrac{[S(x)]^1 \quad \dfrac{\forall x(S(x) \to H(x))}{S(x) \to H(x)}\ \forall E}{H(x)}\ \to E}{\exists x H(x)}\ \exists I}{\exists x H(x)}\ \exists E^1$$

What the derivation accomplishes should be plausible. From the premise that there is an expert skier and the additional premise that all expert skiers are human beings, the derivation establishes that there is a human being.

It is important to note that rule $\exists E$ can only be applied provided a restriction is satisfied, repeated here for your convenience:

RESTRICTION ON $\forall I$. Variables x does not occur free in ψ and x does not occur free in any uncanceled assumption in the sub-derivation of ψ except for $\varphi(x)$.

To understand the reason behind the restriction, let's see a couple of examples of violations. Consider the following derivation (which is a slight variant of the above derivation):

$$\dfrac{\exists x S(x) \qquad \dfrac{[S(x)]^1 \qquad \dfrac{\forall x(S(x) \to H(x))}{S(x) \to H(x)}\,\forall E}{H(x)}\,\to E}{H(x)}\ \textbf{WRONG!}$$

The last step is wrong because it violates the restrictions on $\exists E$. Variable x is free in $H(x)$. Why is this a wrong step? Among other things, note that one could go on and apply the rule for the introduction of the universal quantifier and conclude that all x are human beings, i.e. $\forall x H(x)$. The derivation would look like this:

$$\dfrac{\dfrac{\exists x S(x) \qquad \dfrac{[S(x)]^1 \qquad \dfrac{\forall x(S(x) \to H(x))}{S(x) \to H(x)}\,\forall E}{H(x)}\,\to E}{H(x)}\ \textbf{WRONG!}}{\forall H(x)}\,\forall I$$

It is clearly wrong to conclude that $\forall x H(x)$ from $\exists x S(x)$ and $\forall x(S(x) \to H(x))$. Even if all skiers are human beings and if someone is a skier, it does not follow that everybody is a human being. So, the above derivation must be wrong. But what's wrong in the above derivation is not the application of $\forall I$. What's wrong in the above derivation is the misapplication of $\exists E$.

Here is a second example of a misapplication of $\exists E$ (which consists of a slight modification of the above derivation):

$$\dfrac{\exists x S(x) \qquad \dfrac{\dfrac{[S(x)]^1 \qquad S(x) \to H(x)}{H(x)}\,\to E}{\exists x H(x)}\,\exists I}{\exists x H(x)}\ \textbf{WRONG!}$$

The problem here is that the sub-derivation of $\exists x H(x)$ depends on an uncanceled assumption in which x occurs free, namely $S(x) \to H(x)$. Note that the problem is not with $S(x)$ *per se*, but with $S(x) \to H(x)$. The problem here is that although the formula $\exists x S(x)$ does not say exactly which individual satisfies S, the derivation of $\exists x H(x)$ rests on the additional assumption $S(x) \to H(x)$ which is about x in particular.

Looking at the above (wrong) derivation, we should ask: can we safely say that we are deriving the conclusion $\exists x H(x)$ from $\exists x S(x)$ together with $S(x) \to H(x)$ *and nothing else*? In other words, have we really derived $\exists x H(x)$ from $\exists x S(x)$ and $S(x) \to H(x)$ alone, or have we derived $\exists x H(x)$ from stronger premises? In fact, we have used stronger premises. Which stronger premises have we used? We have used the premise that $S(x)$ where—crucially—variable x happens to be the same variable as the variable in the other premise $S(x) \to H(x)$. So, the stronger premise is that we are fixing on one variable x for both $S(x)$ and $S(x) \to H(x)$. And in deriving $\exists x H(x)$, it was crucial that we were fixing on the same x for both $S(x)$ and $S(x) \to H(x)$. For suppose that instead of using $S(x) \to H(x)$, we were using $S(y) \to H(y)$ without using the same variable x across the two premises. If so, our derivation would not go through any longer. All in all, our derivation of $\exists x H(x)$ from $\exists x S(x)$ together with $S(x) \to H(x)$ rests on *something else,* and it is therefore a misapplication of rule $\exists E$.

In short, there are two ways to violate the restriction on $\exists E$. One violation is that variable x occurs free in the formula ψ that we are attempting to derive from $\exists x \varphi(x)$. Another violation is that variable x occurs free in some uncanceled assumption (except $\varphi(x)$ itself) on which the sub-derivation of ψ depends.

6 EXAMPLES

Let's now see some examples of derivations involving rules $\forall E$, $\forall I$, $\exists I$, and $\exists E$, together with the rules from propositional logic. The expression $\vdash \varphi$ means that there is a derivation of φ from no uncanceled assumption.

$\vdash \forall x \forall y R(x,y) \to \forall y \forall x R(x,y)$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\forall x \forall y R(x,y)]^1}{\forall y R(x,y)}\,\forall E}{R(x,y)}\,\forall E}{\forall x R(x,y)}\,\forall I}{\forall y \forall x R(x,y)}\,\forall I}{\forall x \forall y R(x,y) \to \forall y \forall x R(x,y)}\,\to I^1$$

$\vdash \forall x(A(x) \wedge B(x)) \to \forall x(A(x)) \wedge \forall x(B(x))$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\forall x(A(x) \wedge B(x))]^1}{A(x) \wedge A(x)}\,\forall E}{A(x)}\,\wedge E}{\forall x(A(x))}\,\forall I \qquad \cfrac{\cfrac{\cfrac{[\forall x(A(x) \wedge B(x))]^1}{A(x) \wedge B(x)}\,\forall E}{B(x)}\,\wedge E}{\forall x(B(x))}\,\forall I}{\forall x(A(x)) \wedge \forall x(B(x))}\,\wedge I}{\forall x(A(x) \wedge B(x)) \to \forall x(A(x)) \wedge \forall x(B(x))}\,\to I^1$$

$\vdash \forall x(\varphi(x) \to \psi(x)) \to (\forall x\varphi(x) \to \forall x\psi(x))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\varphi(x)\to\psi(x))]^1}{\varphi(x)\to\psi(x)}\;\forall E \qquad \dfrac{[\forall x\varphi(x)]^2}{\varphi(x)}\;\forall E}{\psi(x)}\;\to E}{\forall x\psi(x)}\;\forall I}{\forall x\varphi(x)\to\forall x\psi(x)}\;\to I^2}{\forall x(\varphi(x)\to\psi(x))\to(\forall x\varphi(x)\to\forall x\psi(x))}\;\to^1$$

$\vdash \forall x\varphi(x) \to \neg\forall x\neg\varphi(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\varphi(x)]^1}{\varphi(x)}\;\forall E \qquad \dfrac{[\forall x\neg\varphi(x)]^2}{\neg\varphi(x)}\;\forall E}{\bot}\;\to E}{\neg\forall x\neg\varphi(x)}\;\to I^2}{\forall x\varphi(x)\to\neg\forall x\neg\varphi(x)}\;\to I^1$$

$\vdash \neg\forall x\varphi(x) \to \exists x\neg\varphi(x)$

$$\dfrac{\dfrac{\dfrac{[\neg\forall x\varphi(x)]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg\exists x\neg\varphi(x)]^2 \qquad \dfrac{[\neg\varphi(x)]^3}{\exists x\neg\varphi(x)}\;\exists I}{\bot}\;\to E}{\varphi(x)}\;RAA^3}{\forall x\varphi(x)}\;\forall I}{\bot}\;\to E}{\exists x\neg\varphi(x)}\;RAA^2}{\neg\forall x\varphi(x)\to\exists x\neg\varphi(x)}\;\to I^1$$

$\vdash \exists x\neg\varphi(x) \to \neg\forall x\varphi(x)$

$$\dfrac{\dfrac{\dfrac{[\exists x\neg\varphi(x)]^1 \qquad \dfrac{[\neg\varphi(x)]^3 \qquad \dfrac{[\forall x\varphi(x)]^2}{\varphi(x)}\;\forall E}{\bot}\;\to E}{\bot}\;\exists E^3}{\neg\forall x\varphi(x)}\;\to I^2}{\exists x\neg\varphi(x)\to\neg\forall x\varphi(x)}\;\to I^1$$