PHIL 50 – INTRODUCTION TO LOGIC

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HOMEWORK – WEEK #10 – SOLUTIONS

1 PROVING THINGS WITH THE PROBABILITY AXIOMS [40 POINTS]

Using the probability axioms, the definition of conditional probability, and the theorems NEGATION, EQUIVALENCE, and TOTAL PROBABILITY, show the following:

(a) If $\varphi \models \psi$, then $P(\varphi) \leq P(\psi)$

(b) $P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\varphi \wedge \psi)$

SOLUTION to part (a).

1. Suppose $\varphi \models \psi$
2. So, from 1, it follows that $\varphi \models \varphi \wedge \psi$
3. It holds that $\varphi \wedge \psi \models \varphi$
4. From 2 and 3, it follows that the formulas φ and $\varphi \wedge \psi$ are logically equivalent
4. From 4 and by EQUIVALENCE, we have that $P(\varphi) = P(\varphi \wedge \psi) = P(\psi \wedge \varphi)$
5. $P(\psi \wedge \varphi) = P(\psi)P(\varphi|\psi)$, from the definition of conditional probability
6. So, from 4 and 5, it follows that $P(\varphi) = P(\psi)P(\varphi|\psi)$
7. From 6, by algebra, it follows that $\frac{P(\varphi)}{P(\varphi|\psi)} = P(\psi)$
8. If $P(\varphi|\psi) = 1$, then $P(\psi) = P(\varphi)$, and if instead $P(\varphi|\psi) < 1$, then $P(\varphi) < P(\psi)$

SOLUTION to part (b).

1. It easy to check with a truth table that $\varphi \vee \psi \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)) \vee (\neg\varphi \wedge \psi)$
2. From 1 and EQUIVALENCE, $P(\varphi \vee \psi) = P(((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)) \vee (\neg\varphi \wedge \psi))$
3. By ADDITIVITY, $P(((\varphi\wedge\psi)\vee(\varphi\wedge\neg\psi))\vee(\neg\varphi\wedge\psi)) = P((\varphi\wedge\psi)\vee(\varphi\wedge\neg\psi))+P(\neg\varphi\wedge\psi)$
4. By ADDITIVITY, $P(((\varphi\wedge\psi)\vee(\varphi\wedge\neg\psi))\vee(\neg\varphi\wedge\psi)) = P(\varphi\wedge\psi)+P(\varphi\wedge\neg\psi)+P(\neg\varphi\wedge\psi)$
5. From 2, 3, and 4, we have $P(\varphi \vee \psi) = P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi) + P(\neg\varphi \wedge \psi)$

6a. $P(\varphi \wedge \psi) = P(\psi \wedge \varphi) = P(\psi)P(\varphi|\psi)$

6b. $P(\varphi \wedge \neg\psi) = P(\neg\psi \wedge \varphi) = P(\neg\psi)P(\varphi|\neg\psi)$

6c. $P(\neg\varphi \wedge \psi) = P(\neg\varphi)P(\psi|\neg\varphi)$

7. From 5 and 6, we have $P(\varphi \vee \psi) = P(\psi)P(\varphi|\psi) + P(\neg\psi)P(\varphi|\neg\psi) + P(\neg\varphi)P(\psi|\neg\varphi)$
8. By rearranging and by adding and subtracting $P(\psi|\varphi)P(\varphi)$ in equation in 7, we have

$$P(\varphi \vee \psi) = P(\varphi|\psi)P(\psi) + P(\varphi|\neg\psi)P(\neg\psi) + P(\psi|\varphi)P(\varphi) + P(\psi|\neg\varphi)P(\neg\varphi) - P(\psi|\varphi)P(\varphi)$$

9a. By TOTAL PROBABILITY, $P(\varphi) = P(\varphi|\psi)P(\psi) + P(\varphi|\neg\psi)P(\neg\psi)$
9b. By TOTAL PROBABILITY, $P(\psi) = P(\psi|\varphi)P(\varphi) + P(\psi|\neg\varphi)P(\neg\varphi)$
10. From 8 and 9, we have $P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\psi|\varphi)P(\varphi)$.
11. From 10, we have $P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\varphi \wedge \psi)$ b/c $P(\varphi \wedge \psi) = P(\varphi)P(\psi|\varphi)$.

2 BAYES [30 POINTS]

Imagine that, in a small town, there are two bus companies, GREEN and BLUE, whose buses are respectively painted green and blue. GREEN company covers 85 percent of the market and BLUE company covers the rest. There are no other companies around. On a misty day, a bus hits and injures a passerby, but it drives off.

A witness reports that it was a blue bus. The witness is right only 80 percent of the time. This means that he gets the color right 80 percent of the time. More formally, this means

$0.8 = Pr($*witness says the bus is blue*$|$*the bus is blue*$)$, and also

$0.8 = Pr($*witness says the bus is green*$|$*the bus is green*$)$.

Given the witness report, what is the probability that the taxi cab involved in the accident was in fact blue? Hint: Use Bayes' rule and the rule of total probability.

ANSWER.

W_b: *The witness says that the bus is blue*
W_g: *The witness says that the bus is green*
B: *The bus is blue*
G: *The bus is green*

The probability that we want to know is $Pr(B|W_b)$. The key here is to use Bayes' rule:

$$Pr(B|W_b) = \frac{Pr(W_b|B) \times Pr(B)}{Pr(W_b|B) \times Pr(B) + Pr(W_b|G) \times Pr(G)}.$$

The denominator is the rule of total probability. From the data of the problem, we have:

$$Pr(B|W_b) = \frac{0.80 \times 0.15}{0.80 \times 0.15 + 0.2 \times 0.58} \approx 0.41.$$

This should be self-explanatory. Note that $Pr(W_b|G) = 0.2$. Why? The reason is that $Pr(W_g|G) = 0.8$, so by the negation rule, we have $Pr(W_b|G) = 0.2$.

3 IS PROBABILITY TRUTH-FUNCTIONAL? [30 POINTS]

In propositional logic, once you know the truth values of the atomic propositions, you can know the truth values of any arbitrarily complex formula containing such atomic propositions. So, if you know the truth values of p, q and r, it is straightforward to determine the truth value of, say, $p \vee (\neg q \wedge \neg(q \wedge r))$. Does something similar hold for probability assignments? In other words, is it the case that if you know the probability of p and q, then you'll know the probability of $p \vee q$, or the probability of $p \wedge q$, or the probability of any arbitrarily complex formula built out of p and q?

(a) Check whether if you know the probability of p and q, then you know the probability of $p \wedge q$. If not, please provide a counterexample. The counterexample will be such that, given a certain assignment of probability for p and q, there are multiple probability assignments for $p \wedge q$ that are all compatible with the probability axioms.

(b) Do the same as with part (a), but use $p \vee q$ instead of $p \wedge q$.

(c) Do the same as with parts (a) and (b), but use $\neg p$.

SOLUTION to part (a). Here is a counterexample. Suppose $P(p) = 0.5$, $P(q) = 0.5$, and $P(p \wedge q) = 0.25$. This is certainly possible given the probability axioms. In fact, we know that $P(p \wedge q) = P(p)P(q|p)$ from the definition of conditional probability. We can assume that $P(q|p) = P(q)$, so $P(p \wedge q) = P(p)P(q) = 0.5 \times 0.5 = 0.25$. But another scenario is also possible. Suppose $P(p) = 0.5$, $P(q) = 0.5$, and $P(p \wedge q) = 0.5$ by assuming that $P(q|p) = 1$, so hat $P(p \wedge q) = P(p)P(q|p) = 0.5 \times 1 = 0.5$. All in all, once you know the probability of p and q, you do not yet know the probability of $p \wedge q$. You do need to know the conditional probability $P(q|p)$.

SOLUTION to part (b). We get the same result as in part (a). Note that from exercise 1(b), it follows that $P(p \vee q) = P(p) + P(q) - P(p \wedge q)$. In part (a) of this exercise, we showed that the probability of q and q is not enough to determine the probability of $p \wedge q$, and thus it is not enough to determine the probability of $p \vee q$ either. Of course, it is true that by ADDITIVITY, $P(p \vee q) = P(p) + P(q)$, but that holds provided p and q form a contradiction, which is not the case.

SOLUTION to part (c). We can see that if we know the probability of p, we know the probability of $\neg p$. The NEGATION rule makes this straightforward to see because $P(p) = 1 - P(\neg p)$.