PHIL 50 – INTRODUCTION TO LOGIC

MARCELLO DI BELLO – STANFORD UNIVERSITY

HOMEWORK – WEEK #2 – SOLUTIONS

1 FORMULAS [10 POINTS]

Check whether the following are formulas of our propositional language:

$(\varphi \wedge \psi) \rightarrow \neg(\varphi \vee (\varphi \vee \varphi))$

$\neg\neg\neg \rightarrow \varphi$

ANSWER. Neither is a formula because φ and ψ are schemata for formulas, not real formulas themselves. But if we were to treat φ and ψ as actual formulas, the first would be a well-formed formula, but not the second. To establish that something is a formula, it is enough to construct it by means of the different clauses in the inductive definition of a formula. To show that something is not a formula, it is enough to show that it conforms to no one of the clauses in the inductive definition of a formula.

2 INDUCTIVE DEFINITIONS [20 POINTS]

(a) Give an inductive definition of the number of brackets for the formulas in the propositional language. As inductive cases, consider only $\neg$ and $\wedge$.

(b) Give an inductive definition of the function that assigns truth values (1 and 0) to the formulas in the propositional language. As inductive cases, consider only $\neg$ and $\wedge$.

ANSWER to part (a). Let $\#b[\varphi]$ denote the function that gives us the number of brackets for a formula φ. This function can be defined inductively as follows. The BASE case is $\#b[p] = 0$, where p is an atomic formula. Let's consider two INDUCTIVE cases, one for $\wedge$ and one for $\neg$. As for $\wedge$, we have $\#b[\varphi \wedge \psi] = \#b[\varphi] + \#b[\psi] + 2$. As for $\neg$, we have $\#b[\neg\varphi] = \#b[\varphi]$.

ANSWER to part (b). Let $[[\varphi]]$ denote the function that assigns truth values to all formulas φ in the propositional language. The BASE case is $[[p]] = V(p)$, where p is an atomic formula and V denotes the function that assign to every atomic formula a truth value. Let's consider two INDUCTIVE cases, one for $\wedge$ and one for $\neg$. As for $\wedge$, we have $[[\varphi \wedge \psi]] = min([[\varphi]], [[\psi]])$. As for $\neg$, we have $[[\neg\varphi]] = 1 - [[\varphi]]$.

3 IF....THEN [20 POINTS]

You have learned that statements of the form $\varphi \rightarrow \psi$ are (vacuously) true whenever φ is false. But this might be different from the ordinary meaning we attribute to statements of the form *if...then*.

(a) Collect two examples of natural language statements of the form *if...then* which do not (seem to) conform to the material conditional $\rightarrow$.[1]

[1] Here is an example: *If the US economy collapses, the world economy collapses.* Now, as of now the US economy has not (yet) collapsed, so the antecedent of this *if...then* statement is false. But the mere fact that the antecedent is false

(b) Does the connective *if...then* which occurs in the examples you have collected in point (a) above behave truth-functionally or not?

Explain your answers.

ANSWER to part (a). One example is *If I were you, I would leave this place and get a job.* This does not conform to the material implication because the mere fact that I am not you does not make the *if...then* statement in question true. Another example is *If I do not get 8 hours of sleep every day, I cannot think during the day*. Again, the mere fact that I do not sleep 8 hours every day does not make the *if...then* statements in question true.

ANSWER to part (b). In both cases, there is a close connection between if-statement and then-statement, and this connection cannot be captured by their truth-value arrangements. In order for the *if-then* statements in (a) to be true, it does not matter whether the constituent if- and then-statements are true or false; what matters is their connection, i.e. whether supposing that the if-statement were true, it is the case that the then-statement is also true. So, the *if...then* connectives in the examples in (a) are not used truth-functionally.

4 TRUTH-FUNCTIONS AND LOGICAL CONSEQUENCE [15 POINTS]

(a) Write a mathematical function corresponding to the meaning of the connective $\rightarrow$.

(b) Let $\{\varphi_1, \varphi_2, \varphi_3, \dots\}$ be an infinite set of formulas. Can you use truth-tables to check whether the following holds?

$$\{\varphi_1, \varphi_2, \varphi_3, \dots\} \models \psi$$

If not, why not?

(c) Suppose you know that the following holds (with p_1 and p_2 atomic formulas):

$$\{p_1, p_2\} \models \psi$$

Does it follow that the following also holds?

$$\{p_1, p_2, \neg\psi\} \models \psi$$

(d) Let $\{p_1, p_2, p_3, \dots\}$ be an finite set of atomic formulas. Can you use truth-tables to check whether the following holds?

$$\{p_1, p_2, p_3, \dots\} \models p_1 \rightarrow p_3$$

Explain your answers.

ANSWER to part (a). A mathematical function corresponding to the meaning of the connective $\rightarrow$ is the two-place function $(1 - x_1) + (x_1 x_2) = y$, where x_1 and x_2 are the inputs and y is the output. If $x_1 = x_1 = 1$, then $y = 1$. If $x_1 = x_1 = 0$, then $y = 1$. If $x_1 = 1$ and $x_2 = 0$, then $y = 0$. If $x_1 = 0$ and $x_2 = 1$, then $y = 1$. This corresponds to $\rightarrow$.

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does not make the entire statement true in so far as ordinary language is concerned. Since the statement in question does not seem (vacuously) true, then we may conclude that the statement does not contain a material conditional, but some other connective.

ANSWER to part (b). No. It is not clear what the relation between the formulas in $\{\varphi_1, \varphi_2, \varphi_3, \dots\}$ and ψ is. After all, we are dealing with schemata of formulas, not with real formulas. We need to know more about the structure of each formula to check whether $\{\varphi_1, \varphi_2, \varphi_3, \dots\} \models \psi$ holds.

ANSWER to part (c). It follows. We should check whether $\{p_1, p_2, \neg\psi\} \models \psi$ holds. In order to do that, we should establish that whenever p_1, p_2 and $\neg\psi$ are all true, ψ is also true. So, let's suppose that p_1, p_2 and $\neg\psi$ are all true. It follows that both ψ and $\neg\psi$ are true. (This holds because by assumption we have that $\{p_1, p_2\} \models \psi$, so whenever both p_1 and p_2 are true, then ψ is also true.) But it is impossible for both ψ and $\neg\psi$ to be true, because ψ is true if and only if $\neg\psi$ is false. So, p_1, p_2 and $\neg\psi$ cannot be all true. If they cannot be all true, then $\{p_1, p_2, \neg\psi\} \models \psi$ holds vacuously. (Recall that $\{p_1, p_2, \neg\psi\} \models \psi$ has the form of an implication. i.e. if $p_1, p_2, \neg\psi$ are all true, then ψ is always true. But I have just shown that the antecedent of this implication cannot hold, so the implication holds vacuously, whence $\{p_1, p_2, \neg\psi\} \models \psi$.)

ANSWER to part (d). Yes. The set of formulas $\{p_1, p_2, p_3, \dots\}$ contains both p_1 and p_3. Whenever both these formulas are true, by the truth table for the implication, it must be the case that $p_1 \to p_3$ is also true, whence $\{p_1, p_3\} \models p_1 \to p_3$. But how do we know that from the larger set $\{p_1, p_2, p_3, \dots\}$, the formula $p_1 \to p_3$ still logically follows? Suffices to note that the truth value of $p_1 \to p_3$ depends only on the truth value of p_1 and p_3, regardless of other atomic formulas in $\{p_1, p_2, p_3, \dots\}$. (This is a result of what in lecture I called *logical atomism*.)

5 BIVALENCE SURRENDERED[15 POINTS]

We have seen that the Principle of Non-Contradiction $\neg(\varphi \wedge \neg\varphi)$ and the Principle of Excluded Middle $(\varphi \vee \neg\varphi)$ are valid provided we maintain the Principal of Bivalence. What happens if we do away with it? Let's suppose that formulas can be assigned three truth values, namely 1, 0, and 0.5. And let's suppose that negation, conjunction and disjunction behave as follows:

$\neg$	φ
0	1
1	0
0.5	0.5

φ	$\wedge$	ψ
1	1	1
1	0.5	0.5
1	0	0
0.5	0.5	1
0.5	0.5	0.5
0.5	0	0
0	0	1
0	0	0.5
0	0	0

φ	$\vee$	ψ
1	1	1
1	1	0.5
1	1	0
0.5	1	1
0.5	0.5	0.5
0.5	0.5	0
0	1	1
0	0.5	0.5
0	0	0

(a) Write three mathematical functions, each corresponding to the behavior of the three connectives as defined by three truth tables above.

(b) Use the truth table method to check whether or not the Principle of Non-Contradiction and the Principle of Excluded Middle are still valid.

(c) Use the truth table method to check whether or not the Principle of Non-Contradiction and the Principle of Excluded Middle are still equivalent.

ANSWER to part (a). The three functions are $1 - x = y$; $min(x_1, x_2) = y$; and $max(x_1, x_2) = y$.

ANSWER to part (b) and (c). Neither principle is valid though they are equivalent as the following truth table shows:

$\neg$	$(\varphi$	$\wedge$	$\neg$	$\varphi)$	$(\varphi$	$\vee$	$\neg$	$\varphi)$
1	1	0	0	1	1	1	0	1
1	0	0	1	0	0	1	1	0
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

6 ODD WAYS TO SAY SIMPLE THINGS [20 POINTS]

Suppose that $\bot$ is the connective that is always false, i.e it always gets assigned the value 0. (Strictly speaking $\bot$ is not a connective because it does not connect anything; it just is always false.) Now, consider a propositional language whose connectives are simply $\rightarrow$ and $\bot$. Find ways to write formulas equivalent to $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\neg\varphi$ just by using the connectives $\rightarrow$ and $\bot$. Check you answers using truth tables.

ANSWER. It is easy to see that $\neg\varphi$ is equivalent to $\varphi \rightarrow \bot$:

φ	$\rightarrow$	$\bot$	$\neg$	φ
1	0	0	0	1
0	1	0	1	0

We can also see that $(\varphi \rightarrow \bot) \rightarrow \psi$ is equivalent to $\varphi \vee \psi$:

$(\varphi$	$\rightarrow$	$\bot)$	$\rightarrow$	ψ	φ	$\vee$	ψ
1	0	0	1	1	1	1	1
1	0	0	1	0	1	1	0
0	1	0	1	1	0	1	1
0	1	0	0	0	0	0	0

Finally, we can see that $(\varphi \rightarrow (\psi \rightarrow \bot)) \rightarrow \bot$ is equivalent to $\varphi \wedge \psi$:

$(\varphi$	$\rightarrow$	$(\psi$	$\rightarrow$	$\bot))$	$\rightarrow$	$\bot$	φ	$\wedge$	ψ
1	0	1	0	0	1	0	1	1	1
1	1	0	1	0	0	0	1	0	0
0	1	1	0	0	0	0	0	0	1
0	1	0	1	0	0	0	0	0	0