PHIL 50 – INTRODUCTION TO LOGIC

MARCELLO DI BELLO – STANFORD UNIVERSITY

HOMEWORK – WEEK #3 – SOLUTIONS

1 IMPLICATIONS

Construct derivations for the following formulas:

(a) $\varphi \to (\psi \to (\varphi \wedge \psi))$

(b) $(\varphi \to (\psi \to \sigma)) \to (\psi \to (\varphi \to \sigma))$

(c) $((\varphi \to \psi) \to (\varphi \to \sigma)) \to (\varphi \to (\psi \to \sigma))$

This group of derivation should make you familiar with the rules for $\wedge$ and $\to$. You will use rules '$\to I$' and '$\to E$' a lot.

SOLUTION to (a):

$$\cfrac{\cfrac{\cfrac{[\varphi]^1 \quad [\psi]^2}{\varphi \wedge \psi}\ \wedge I}{\psi \to (\varphi \wedge \psi)}\ \to I^2}{\varphi \to (\psi \to (\varphi \wedge \psi))}\ \to I^1$$

SOLUTION to (b):

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[(\varphi \to (\psi \to \sigma))]^1 \quad [\varphi]^2}{\psi \to \sigma}\ \to E \qquad [\psi]^3}{\sigma}\ \to E}{\varphi \to \sigma}\ \to I^2}{\psi \to (\varphi \to \sigma)}\ \to I^3}{(\varphi \to (\psi \to \sigma)) \to (\psi \to (\varphi \to \sigma))}\ \to I^1$$

SOLUTION to (c):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\varphi\to\psi)\to(\varphi\to\sigma)]^1 \quad \dfrac{\dfrac{[\varphi]^2}{[\psi]^3}}{\varphi\to\psi}\to I^2}{\varphi\to\sigma}\to E \qquad [\varphi]^4}{\sigma}\to E}{\psi\to\sigma}\to I^3}{\varphi\to(\psi\to\sigma)}\to I^4}{((\varphi\to\psi)\to(\varphi\to\sigma))\to(\varphi\to(\psi\to\sigma))}\to I^1
$$

2 DOUBLE IMPLICATION

Consider the formula $((p \to q) \leftrightarrow p) \to q$.

(a) Come up with an informal argument that motivates why the formula is true.

(b) Construct a derivation for the given formula. [*Note that there is no derivation rule for the symbol* $\leftrightarrow$*, so when you encounter a formula containing that symbol just unpack it. Your derivation at some point will look like this:*

$$
\begin{array}{c}
\vdots \\
\dfrac{(p \to q) \leftrightarrow p}{((p \to q) \to p) \land (p \to (p \to q))}\ \text{unpack} \leftrightarrow \\
\vdots
\end{array}
$$

SOLUTION to (a): The antecedent of the formula says that (i) if p implies q, then p follows and that (ii) if p holds, then p implies q. There are two cases. If p is true, by (ii) p implies q, whence q holds. On the other hand, if p is false, then p vacuously implies q, and by (i) this would imply that p holds, but this a contradiction, so p cannot be false. But again, if p is true, then by (ii), q follows. So, q follows no matter the truth value of p. So, given the antecedent, q follows, and this is what the formula in question says, so the formula is aways true.

SOLUTION to (b):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(p\to q)\leftrightarrow p]^1}{((p\to q)\to p)\land(p\to(p\to q))}\leftrightarrow}{p\to(p\to q)}\land E \quad [p]^2}{p\to q}\to E \quad [p]^2}{q}\to E}{p\to q}\to I^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(p\to q)\leftrightarrow p]^1}{((p\to q)\to p)\land(p\to(p\to q))}\leftrightarrow}{p\to(p\to q)}\land E \quad [p]^2}{p\to q}\to E \quad [p]^2}{q}\to E}{p\to q}\to I^2 \quad \dfrac{\dfrac{[(p\to q)\leftrightarrow p]^1}{((p\to q)\to p)\land(p\to(p\to q))}\leftrightarrow}{(p\to q)\to p}\land E}{p}\to E}{q}\to E}{((p\to q)\leftrightarrow p)\to q}\to I^1$$

3 MORE DERIVATIONS

Construct derivations for the following formulas:

(a) $(\varphi\to\psi)\to\neg(\varphi\land\neg\psi)$

(b) $\neg(\varphi\land\neg\psi)\to(\varphi\to\psi)$

(c) $\neg(\varphi\lor\psi)\to(\neg\varphi\land\neg\psi)$

These derivation should make you familiar with with the other derivation rules.

(d) Which one among the derivations you have offered in exercise 3 is the intuitionistic logician unlikely to accept? What does this tell you about the inter-definability of the connectives in intuitionistic logic? Explain.

SOLUTION to (a):

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi\to\psi]^2 \quad \dfrac{[\varphi\land\neg\psi]^1}{\varphi}\land E}{\psi}\to E \quad \dfrac{[\varphi\land\neg\psi]^1}{\neg\psi}\land E}{\bot}\to E}{\neg(\varphi\land\neg\psi)}\to I^1}{(\varphi\to\psi)\to\neg(\varphi\land\neg\psi)}\to I^2$$

SOLUTION to (b):

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi\land\neg\psi)]^3 \quad \dfrac{[\varphi]^1 \quad [\neg\psi]^2}{\varphi\land\neg\psi}\land I}{\bot}\to E}{\psi}\,RAA^2}{\varphi\to\psi}\to I^1}{\neg(\varphi\land\neg\varphi)\to(\varphi\to\psi)}\to I^3$$

SOLUTION to (c):

$$\dfrac{\dfrac{\dfrac{[\neg(\varphi \vee \psi)]^1 \quad \dfrac{[\varphi]^2}{\varphi \vee \psi}\vee I}{\bot}\to E}{\neg\varphi}\to I^2 \quad \dfrac{\dfrac{[\neg(\varphi \vee \psi)]^1 \quad \dfrac{[\psi]^3}{\varphi \vee \psi}\vee I}{\bot}\to E}{\neg\psi}\to I^3}{\neg(\varphi \vee \psi) \to (\neg\varphi \wedge \neg\psi)}\to I^1$$

SOLUTION to (d): The intuitionistic logician is going to accept all derivations except (b) because this rests on the rule RAA. While in classical logic $\neg(\varphi \wedge \neg\psi)$ is equivalent to $(\varphi \to \psi)$, so that $\to$ is definable in terms of $\neg$ and $\wedge$, this is not the case in intutionistic logic.

4 DISJUNCTIVE SYLLOGISM

Disjunctive syllogism is a derivation rule that looks like this:

$$\dfrac{\varphi \vee \psi \quad \neg\varphi}{\psi}\ DS$$

(a) There is no need to add rule DS to our derivation rules, however. For it is possible to derive DS from the rules we have. To this end, construct a derivation establishing that $\varphi \vee \psi, \neg\varphi \vdash \psi$.

(b) Consider the following formulas associated with statements in natural language about a murder case:

w : Mrs White is guilty

s : Miss Scarlet is guilty

m : Colonel Mustard is guilty

$s \vee (w \vee m)$: At least one of them is guilty

$w \to m$: If Mrs White si guilty, so is Colonel Mustard

$\neg s \to \neg m$: If Miss Scarlet is innocent, then so is Colonel Mustard

Using DS and some of the other derivation rules you've learned, construct a derivation establishing that

$$s \vee (w \vee m), w \to m, \neg s \to \neg m \vdash s$$

SOLUTION to (a):

$$\dfrac{[\varphi \vee \psi] \qquad \dfrac{\dfrac{[\neg\varphi] \quad [\varphi]^1}{\bot}\to E}{\psi}\bot \qquad \dfrac{[\psi]^1}{\psi}R}{\psi}\vee E^1$$

SOLUTION to (b):

$$\dfrac{\dfrac{\dfrac{[s \vee (w \vee m)] \quad [\neg s]^1}{w \vee m}DS \quad \dfrac{[\neg s \to \neg m] \quad [\neg s]^1}{\neg m}\to E}{w}DS \qquad \dfrac{\dfrac{\dfrac{[\neg s \to \neg m] \quad [\neg s]^1}{\neg m}\to E \quad \dfrac{[w \to m] \quad [w]^2}{m}\to E}{\bot}\to E}{\neg w}\to I^2}{\dfrac{\bot}{s}RAA^1}\to E$$