PHIL 50 – INTRODUCTION TO LOGIC

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HOMEWORK – WEEK #5 – SOLUTIONS

1 A DERIVATION FOR RUSSELL'S PARADOX [20 POINTS]

We have seen that Russell's paradox amounts to the claim that if $R \in R$, then $R \notin R$, and if $R \notin R$, then $R \in R$. This looks like a contradiction. But is it really? Let's prove it! We can denote the statement $R \in R$ by φ and the statement $R \notin R$ by $\neg\varphi$. Now Russell's paradox boils down to the pair of statements $\varphi \to \neg\varphi$ and $\neg\varphi \to \varphi$. This exercise asks you to derive a contradiction from these two statements. In other words, please show (syntactically) that

$$\varphi \to \neg\varphi, \neg\varphi \to \varphi \vdash \bot$$

SOLUTION.

$$\dfrac{\varphi \vee \neg\varphi \qquad \dfrac{[\varphi]^1 \qquad \dfrac{[\varphi]^1 \quad [\varphi \to \neg\varphi]}{\neg\varphi}\to E}{\bot}\to E \qquad \dfrac{[\neg\varphi]^1 \qquad \dfrac{[\neg\varphi]^1 \quad [\neg\varphi \to \varphi]}{\varphi}\to E}{\bot}}{\bot}\vee E^1$$

Note that the derivation hinges upon $\varphi \vee \neg\varphi$, which is the principle of excluded middle. We con construct an independent derivation of this principle (see the lecture notes on derivations for that).

2 DEVIL'S LOGIC [20 POINTS]

Suppose God wants to pardon you and let you enter the gates of Heaven. And suppose that you—just like all of us—have willfully sinned at some point in your life. So, here is Devil's argument against you being pardoned.

All pardoned people are penitent.
No willful sinner is penitent.

No willful sinner is pardoned.[1]

(a) Translate the syllogism in set theoretic terms.

(b) Check whether the syllogism is valid or invalid. If the syllogism is valid, give a set theoretic argument (i.e. the type of argument you find in Monday's or Wednesday's slides). If the syllogism is invalid, give a counterexample.

SOLUTION to (a). Let Par stand for the set of pardoned people, Pen for the set of penitent people, and Sin for the set of willful sinners. In set theoretic terms, we have:

$Par \subseteq Pen$
$Sin \cap Pen = \emptyset$

$Sin \cap Par = \emptyset$

SOLUTION to (b). Suppose for contradiction that $Sin \cap Par \neq \emptyset$, so there is an $a \in Sin \cap Par$, so $a \in Sin$ and $a \in Par$. Since the first premise in the syllogism is that $Par \subseteq Pen$, it follows that $a \in Pen$. So, $a \in Sin$ and $a \in Pen$. This means that $a \in Sin \cap Pen$, so $Sin \cap Pen \neq \emptyset$, which contradicts the second premise in the syllogism. We can conclude that there is no $a \in Sin \cap Par$, so $Sin \cap Par = \emptyset$. The Devil is right...

3 THE EMPTY SET [30 POINTS]

Consider the following syllogism

All ideas are old ideas.
All old ideas are plagiarized.

Some ideas are plagiarized.

Is this syllogism valid? Well, Aristotle would say yes, and modern logicians would say no. Why? They disagree on whether sets can be empty or not. With this in mind, please do the following:

[1]This is a syllogistic reading of a passage from Dante's *Divine Comedy*, *Inferno*, XXVII, 118-120:
No power can the impenitent absolve;
Nor to repent and will at once consist,
By contradiction absolute forbid.

(a) Write the syllogism above in set theoretical terms.

(b) Assume that every set has to be non-empty, so there should be at least one element in every set. Show that the syllogism in question is valid using a set theoretic argument just like the argument that you find in Monday's or Wednesday's slides.

(c) Now drop the assumption that every set has to be non-empty, and construct a counterexample to the above syllogism.

SOLUTION to (a). Let's say that the three predicates in the syllogism refer to the sets I (for Ideas), O (for Old ideas), and P (for Plagiarized). We can write the syllogism set theoretically as follows:

$$\begin{array}{l} I \subseteq O \\ O \subseteq P \\ \hline I \cap P \neq \emptyset \end{array}$$

SOLUTION to (b). Let's assume that there is at least one element a such that $a \in I$. So, $a \in O$ and $a \in P$ because $I \subseteq O$ and $O \subseteq P$. So, $a \in I$ and $a \in P$, whence $I \cap P \neq \emptyset$. The syllogism is thus valid.

SOLUTION to (c). Now, suppose the set I is empty, so $I \subseteq O$. Let's also suppose $O \subseteq P$. Since I is empty, it follows that $I \cap P = \emptyset$ regardless of which elements are in P exactly. This is the needed counterexample. So, the syllogism is invalid.

4 VALUATIONS AND FORMULAS [30 POINTS]

Let Γ and Δ be sets of formulas. Let $Val(\Gamma)$ be the set of valuations which make true all the formulas in Γ. More precisely, $Val(\Gamma) = \{V| \textit{for all } \varphi\textit{, if } \varphi \in \Gamma\textit{, then } V(\varphi) = 1\}$. And similarly for $Val(\Delta)$, so that $Val(\Delta) = \{V| \textit{for all } \varphi\textit{, if } \varphi \in \Delta\textit{, then } V(\varphi) = 1\}$. In other words, Val gives us all the valuations that make true all all the formulas in a given set of formulas, be this set Γ or Δ. Now, please do following:

(a) Construct a counterexample to the claim that if $\Gamma \subseteq \Delta$, then $Val(\Gamma) \subseteq Val(\Delta)$.

(b) Suppose $\Gamma \subseteq \Delta$. Show that $Val(\Delta) \subseteq Val(\Gamma)$.

SOLUTION to (a). Consider the sets $\{p\}$ and $\{p, q\}$. Take a V such that $V(p) = 1$ and $V(q) = 0$. Now, $V \in Val(\{p\})$, but $V \notin Val(\{p, q\})$. So, $\{p\} \subseteq \{p, q\}$, but $Val(\{p\}) \not\subseteq Val(\{p, q\})$.

SOLUTION to (b). Suppose $\Gamma \subseteq \Delta$. We want to show that $Val(\Delta) \subseteq Val(\Gamma)$. In other words, we want to show that for any valuation V, if $V \in Val(\Delta)$, then $V \in Val(\Gamma)$. So, let's assume (**) that $V \in Val(\Delta)$, and let's aim to show that $V \in Val(\Gamma)$. Here is a step by step proof:

1. Assume that $V \in Val(\Delta)$, for an arbitrary V.
2. $V \in Val(\Delta)$ iff [for all φ, if $\varphi \in \Delta$, then $V(\varphi) = 1$] by definition of Val.
3. So, for all φ, if $\varphi \in \Delta$, then $V(\varphi) = 1$, from 1 and 2.
4. So, for all φ, if $\varphi \in \Gamma$, then $V(\varphi) = 1$ because $\Gamma \subseteq \Delta$.
5. $V \in Val(\Gamma)$ iff [for all φ, if $\varphi \in \Gamma$, then $V(\varphi) = 1$] by definition of Val.
6. So, $V \in Val(\Gamma)$, from 4 and 5.
7. Hence, if $V \in Val(\Delta)$, then $V \in Val(\Gamma)$, from 1 and 6.
8. The valuation V was arbitrary, so $Val(\Delta) \subseteq Val(\Gamma)$.