PHIL 50 – INTRODUCTION TO LOGIC

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HOMEWORK – WEEK #6 – SOLUTIONS

1 HEALTHCARE [30 POINTS]

Consider the following argument:

> (1) The Obama's healthcare bill does *not* accomodate everybody's demands, and in contrast, (2) all healthcare bills that do accomodate everybody's demands foster social cohesion. Therefore, (3) the Obama's healthcare bill does not foster social cohesion.

Now, please do the following:

(a) Translate statements (1), (2), and (3), in set-theoretic notation. [*Hint*: the Obama's healthcare bill can simply be treated as one object, call it ob. In addition, let HB be the set of healthcare bills, A the set of what accommodates everybody's needs, and FC the set of what fosters social cohesion. Use operations on sets and set membership to carry out the set-theoretic translation.]

(b) Check whether the argument is valid using a set-theoretic argument or give a counterexample if the argument is invalid.

(c) Translate statements (1), (2), and (3) using the language of predicate logic. [*Hint*: Let ob be the constant symbol for Obama's health care bill; let HB be the predicate symbol for the attribute of being a healthcare bill; let A be the predicate symbol for the attribute of accommodating everybody's needs; let FC be the predicate symbol for the attribute of fostering social cohesion. Use connectives and quantifiers to carry out the translation.]

SOLUTION to (a).

(1) $ob \in HB \cap \overline{A}$
(2) $HB \cap A \subseteq FC$

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(3) $ob \notin FC$

SOLUTION to (b). The argument is not valid. To give a counterexample, we construct a case in which both (1) and (2) are satisfied, but (3) is not. Suppose there is only one element, namely ob. Suppose $ob \in HB$ and $ob \notin A$. Suppose also that $ob \in FC$. It follows that (1) $ob \in HB \cap \overline{A}$ is satisfied because we assumed that $ob \in HB$ and $ob \notin A$, so $ob \in \overline{A}$. Now, (2) $HB \cap A \subseteq FC$ is vacuously satisfied. For $HB \cap A$ is empty given that $ob \notin A$ and ob is the only object. However, (3) $ob \notin FC$ is not satisfied because we assumed that $ob \in FC$.

SOLUTION to (c).

(1) $HB(ob) \land \neg A(ob)$
(2) $\forall x((HB(x) \land A(x)) \to FC(x))$

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(3) $\neg FC(ob)$

2 MISMATCHES [20 POINTS]

Formalize in predicate logic the following chunks of sentences:

(a) Liv likes Ron. Ron likes Debbie. Debbie likes Liv. There is no one who is liked by someone they like. But everyone is liked by someone. And everyone likes someone. But there is no one whom everybody likes.

(b) Some Americans are poor. Some Americans are rich. All Americans are either rich or poor, not both. Some rich Americans give to poor Americans. Some rich Americans do not give to poor Americans. If there were no poor American to whom some rich Americans did not give, then no American would be poor and no American would be rich. (*As for the last sentence, translate its antecedent and consequent separately. Then, put them together by using the material implication. Treat the verbs "were" and "would be" as if they were in the indicative mood, and treat the verb "did not give" as if it were in the present tense.*)

SOLUTION to (a) Let l refer to Liv, r to Ron, d to Debbie. Let $Like$ refer to the relation of liking. Here is the translation of each sentence:

$Like(l, r)$
$Like(r, d)$
$Like(d, l)$
$\neg\exists x\exists y(Like(x, y) \land Like(y, x))$
$\forall x\exists y(Like(y, x))$
$\forall x\exists y(Like(x, y)$
$\neg\exists x\forall y(Like(y, x))$

SOLUTION to (b) Let A refer to the attribute of being American, let $Rich$ refer to the attribute of being rich, and $Poor$ refer to the attribute of being poor. Let $Give$ refer to the relation of giving (money...) to others. Here is the translation of each sentence:

$\exists x(A(x) \land Poor(x))$
$\exists x(A(x) \land Rich(x))$
$\forall x(A(x) \to ((Rich(x) \lor Poor(x)) \land \neg(Rich(x) \land Poor(y))))$
$\exists x(A(x) \land Rich(x)) \land \exists y(A(y) \land Poor(y) \land Give(x, y))$
$\exists x(A(x) \land Rich(x)) \land \exists y(A(y) \land Poor(y) \land \neg Give(x, y))$
$\neg\exists x((A(x) \land Poor(x)) \land \exists y(A(y) \land Rich(y) \land \neg Give(y, x)))$
$\to$
$\neg\exists x(A(x) \land Poor(x)) \land \neg\exists x(A(x) \land Rich(x))$

3 TRUTHS AND ARROWS [35 POINTS]

Consider the following situations:

Situation 1 : $\sharp \rightleftarrows \clubsuit$ *Situation 2* : $\triangle \longrightarrow \heartsuit$

Let's assume that:
- the relation symbol ‘R_1” refers to the arrow-relation in *Situation 1*;
- the relation symbol “R_2” refers to the arrow-relation in *Situation 2*;
- the constants “*sharp*” and “*club-suit*” refer to the objects $\sharp$ and $\clubsuit$ respectively;
- the constants “*triangle*” and “*heart-suit*” refer to the objects $\triangle$ and $\heartsuit$ respectively.

Remark 1: A formula such as $R_1(\textit{sharp}, \textit{club-suit})$ should be understood as saying that the arrow (in *Situation 1*) goes from the object $\sharp$ to the object $\clubsuit$. Similarly, $R_2(\textit{triangle}, \textit{heart-suit})$ should be understood as saying that the arrow (in *Situation 2*) goes from $\triangle$ to $\heartsuit$.

Remark 2: Note an important difference between *Situation 1* and *Situation 2*. Relative to *Situation 1*, the formula $R_1(\textit{sharp}, \textit{club-suit})$ and the formula $R_1(\textit{club-suit}, \textit{sharp})$ are both true, because the arrow goes both directions. Instead, relative to *Situation 2*, the formula $R_2(\textit{triangle}, \textit{heart-suit})$ is true, but the formula $R_2(\textit{heart-suit}, \textit{triangle})$ is not true. The arrow only goes one direction.

Remark 3: When you are checking (relative to *Situation 1*) the truth of a formula with the universal quantifier, e.g. $\forall x R_1(x, \textit{sharp})$, you should consider all objects in *Situation 1*, namely $\sharp$ and $\clubsuit$. Similarly, when you are checking (relative to *Situation 2*) the truth of a formula with the universal quantifier, e.g. $\forall x R_2(x, \textit{triangle})$, you should consider all objects in *Situation 2*, namely $\triangle$ and $\heartsuit$.

Now check whether the following formulas are true relative to the given situation:

(a) $R_1(\textit{club-suit}, \textit{sharp}) \to \neg R_1(\textit{sharp}, \textit{club-suit})$ relative to *Situation 1*

(b) $\neg(R_2(\textit{heart-suit}, \textit{triangle}) \vee \neg R_2(\textit{triangle}, \textit{heart-suit}))$ relative to *Situation 2*

(c) $\exists x(R_1(x, \textit{sharp})) \wedge \exists x(R_1(x, \textit{club-suit}))$ relative to *Situation 1*

(d) $\forall x R_1(x, \textit{sharp})$ relative to *Situation 1*

(e) $\exists x \exists y(R_2(x, y)) \wedge \exists y \exists x(R_2(y, x))$ relative to *Situation 2*

(f) $\forall x \exists y R_1(x, y)$ relative to *Situation 1*

(g) $\exists x \forall y R_1(x, y)$ relative to *Situation 1*

Explain your answers as carefully as possible. In the case of quantified formulas, please rephrase the formula in natural language so that you demonstrate you have understood what the formula means.

SOLUTIONS.

(a) In *Situation 1*, both formulas $R_1(\textit{club-suit}, \textit{sharp})$ and $R_1(\textit{sharp}, \textit{club-suit})$ are true because the arrow goes both directions, so the formula $R_1(\textit{club-suit}, \textit{sharp}) \to \neg R_1(\textit{sharp}, \textit{club-suit})$ is false.

(b) The formula $\neg(R_2(\textit{heart-suit}, \textit{triangle}) \vee \neg R_2(\textit{triangle}, \textit{heart-suit}))$ is equivalent—by propositional logic—to $\neg R_2(\textit{heart-suit}, \textit{triangle}) \wedge R_2(\textit{triangle}, \textit{heart-suit})$. The formula is true in *Situation 2* because both conjuncts are true.

(c) The first conjunct of the formula says that there exists at least one object such that the arrow goes from that object to $\sharp$. This is true in *Situation 1*, because there is such an object, namely $\clubsuit$. The second conjunct of the formula says that there exists at least one object such that the arrows goes from that object to $\clubsuit$. This is true in *Situation 1*, because there is such an object, namely $\sharp$. Since both conjuncts are true, the entire formula is also true in *Situation 1*.

(d) The formula $\forall x R_1(x, \textit{sharp})$ is false relative to *Situation 1*. The formula in question says that from any object the arrow goes to $\sharp$. While it is true that the arrow goes from $\clubsuit$ to $\sharp$, the arrow does not go from $\sharp$ to $\sharp$. So, it is not true that from any object the arrows goes to $\sharp$. So the formula is false in *Situation 1*.

(e) The formula $\exists x \exists y(Rxy) \wedge \exists y \exists x(Ryx)$ is true relative to *Situation 2*. The first conjunct says that there is an object (call it x) from which the arrow goes to an object (call it y). This is certainly true: just take $\triangle$ and $\heartsuit$. The second conjunct says exactly the same thing as what the first conduct says. The second conjunct says that that there is an object (call it y) from which the arrow goes to an object (call it x). This is certainly true: take again $\triangle$ and $\heartsuit$. So, the formula is true true relative to *Situation 2*.

(f) The formula $\forall x \exists y R_1(x, y)$ says that from all the objects the arrow goes to at least one object. This is true in *Situation 1*. Take the objects $\sharp$ and $\clubsuit$. In both cases the arrow goes to at least one object, namely the arrow goes from $\clubsuit$ to $\sharp$ and from $\sharp$ to $\clubsuit$. There are no other objects to consider. So, the formula is true in *Situation 1*.

(g) The formula $\exists x \forall y R_1(x, y)$ says that there is an object from which the arrows goes to every object. This is not true in *Situation 1*. Consider $\sharp$; then, there his no arrow that goes from $\sharp$ to $\sharp$. Also, consider $\clubsuit$; then, there is no arrow that goes from $\clubsuit$ to $\clubsuit$. Since $\sharp$ and $\clubsuit$ are all the objects to consider, we can conclude that there is no object such that the arrow goes from it to any object. Hence, the formula in question is false in *Situation 1*.

4 VAL OF INTERSECTION AND THE INTERSECTION OF VAL [15 POINTS]

This is the same set up as in the last exercise of homework 5. Let Γ and Δ be sets of formulas. Let $Val(\Gamma)$ be the set of valuations which make true all the formulas in Γ. More precisely, $Val(\Gamma) = \{V|$ *for all* φ*, if* $\varphi \in \Gamma$*, then* $V(\varphi) = 1\}$. And similarly for $Val(\Delta)$, so that $Val(\Delta) = \{V|$ *for all* φ*, if* $\varphi \in \Delta$*, then* $V(\varphi) = 1\}$. Now, please do following:

(a) Find a counterexample to the claim that $Val(\Delta \cup \Gamma) = Val(\Delta) \cup Val(\Gamma)$

(b) Show that $Val(\Delta \cup \Gamma) \subseteq Val(\Delta) \cap Val(\Gamma)$

SOLUTION to (a). Let $\Delta = \{p\}$ and let $\Gamma = \{q\}$. Let $V(p) = 1$ and $V(q) = 0$. It follows that $V \in Val(\Delta)$, and thus $V \in Val(\Delta) \cup Val(\Gamma)$, since $Val(\Delta) \subseteq Val(\Gamma) \cup Val(\Delta)$. However, $V \notin Val(\Delta \cup \Gamma)$ because $V(q) = 0$, so V does not make true all the formulas in $\Delta \cup \Gamma$. Hence, the set $Val(\Delta \cup \Gamma)$ and the set $Val(\Delta) \cup Val(\Gamma)$ do not contain the same elements.

SOLUTION to (b). We know that in order to prove that $Val(\Delta \cup \Gamma) \subseteq Val(\Delta) \cap Val(\Gamma)$, we assume that some arbitrary V is such that $V \in Val(\Delta \cup \Gamma)$, and we aim to show that $V \in Val(\Delta) \cap Val(\Gamma)$. Here is a step-by-step proof of the claim:

1. Suppose $V \in Val(\Delta \cup \Gamma)$.
2. $V \in Val(\Delta \cup \Gamma)$ iff [for all φ, if $\varphi \in \Delta \cup \Gamma$, then $V(\varphi) = 1$], by definition of Val.
3. For all φ, if $\varphi \in \Delta \cup \Gamma$, then $V(\varphi) = 1$, from 1 and 2 by logic.
4. For all φ, if $\varphi \in \Delta$, then $V(\varphi) = 1$, from 3 because $\Delta \subseteq \Gamma \cup \Delta$.[1]

[1] You might wonder, how does step 4 follow from step 3 and the fact that $\Delta \subseteq \Gamma \cup \Delta$? Note that step 4 contains an if-then claim. So, in order to establish it, let's assume the antecedent $\varphi \in \Delta$ and let's aim to show the consequent $V(\varphi) = 1$, for some arbitrary formula φ. Now, if $\varphi \in \Delta$, then $\varphi \in \Delta \cup \Gamma$ because $\Delta \subseteq \Gamma \cup \Delta$. By 3, it follows that $V(\varphi) = 1$. Hence, if $\varphi \in \Delta$, then $V(\varphi) = 1$. We proved this for an arbitrary formula φ, so it follows that for all φ, if $\varphi \in \Delta$, then $V(\varphi) = 1$.

5. For all φ, if $\varphi \in \Gamma$, then $V(\varphi) = 1$, from 3 because $\Gamma \subseteq \Gamma \cup \Delta$.
6. $V \in Val(\Delta)$ iff [for all φ, if $\varphi \in \Gamma$, then $V(\varphi) = 1$], by definition of Val.
7. $V \in Val(\Gamma)$ iff [for all φ, if $\varphi \in \Delta$, then $V(\varphi) = 1$], by definition of Val.
8. $V \in Val(\Delta)$, from 4 and 6 by logic.
9. $V \in Val(\Gamma)$, from 5 and 7 by logic.
10. $V \in Val(\Delta)$ and $V \in Val(\Gamma)$, from 8 and 9.
11. $V \in Val(\Delta) \cap Val(\Gamma)$, from 10 by definition of intersection.