PHIL 50 – INTRODUCTION TO LOGIC

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HOMEWORK – WEEK #7 – SOLUTIONS

1 PLANS FOR THE END [10 POINTS]

The plan for the class is to finish up predicate logic during week 8. This will leave us two more weeks. There are some extra topics that could be explored during the final two weeks. Here are some options:

- Intuitionistic logic (i.e. the logic in which the principle of excluded middle is false)
- Paraconsistent logic (i.e. the logic in which the principle that says "from the contradiction anything follows" is false)
- Probability logic (i.e. the logic that explores inductive as opposed to deductive arguments)
- Fuzzy and multi-valued logic (i.e. the logic in which truth and falsity come in degrees, and not simply 1's and 0's)
- Modal logic (i.e. the logic that formalizes statements that are not only about the actual, current, present world, but also statements about possible, imaginary, counterfactual worlds)

We cannot do everything in the last two weeks. Please rank each option from the most desirable to the least desirable. If possible, give a brief motivation for your choices. This will help me prepare for the last two weeks of the course. Thanks very much!

2 WHAT IF "BIRD" MEANT TREE AND "TREE" MEANT BIRD? [30 POINTS]

This exercise invites you to think about whether the words we use actually mean what we think they mean. To this end, consider a simple language whose ingredients are as follows:
- two constant symbols, i.e. "$bird$" and "$tree$";
- two 2-place predicates, i.e. "On" and "$Under$".

First, consider the model $M = \langle D, I, g \rangle$, where:

$D = \{$ , $\}$

The interpretation I for constant symbols and predicate symbols is defined as follows:

$I(bird) =$ 

$I(tree) =$ 

$I(On) = \{\langle$ ,  $\rangle\}$

$I(Under) = \{\langle$ ,  $\rangle\}$

The variable assignment g can be disregarded.

Second, consider the model $M^* = \langle D^*, I^*, g^* \rangle$, where:

$D^* = \{$ ,  $\}$

The interpretation I for constant symbols and predicate symbols is defined as follows:

$I^*(bird) =$ 

$I^*(tree) =$ 

$I^*(On) = \{\langle$ ,  $\rangle\}$

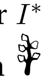
$I^*(Under) = \{\langle$ ,  $\rangle\}$

The assignment g^* can be disregarded.

Now, please do the following:

(a) Check that $M \models On(bird, tree)$ and $M \models Under(tree, bird)$.

(b) Check that $M^* \models On(bird, tree)$ and $M^* \models Under(tree, bird)$.

(c) Suppose the meaning of the words "$bird$" and "$tree$" were determined by an interpretation function such as I or I^*. Given this assumption, while using the word "$bird$", would we be able to tell whether we mean (as per I) or (as per I^*)? And while using the word "$tree$", would we be able to tell whether we mean (as per I) or (as per I^*)? Take into account (a) and (b) above, and respond affirmatively or negatively to these questions. Motivate your answer.

SOLUTION to the first part of (a).

1. $M \models On(bird, tree)$ iff $\langle I(bird), I(tree)\rangle \in I(On)$, by definition of $\models$ in the base case.
2. $\langle I(bird), I(tree)\rangle \in I(On)$ iff $\langle$ ,  $\rangle \in \{\langle$ ,  $\rangle\}$, by definition of $I(bird), I(tree), I(On)$.
3. Since $\langle$ ,  $\rangle \in \{\langle$ ,  $\rangle\}$, from 1 and 2, then $M \models On(bird, tree)$.

SOLUTION to the second part of (a).

1. $M \models Under(tree, bird)$ iff $\langle I(tree), I(bird)\rangle \in I(Under)$, by def. of $\models$.
2. $\langle I(tree), I(bird)\rangle \in I(under)$ iff $\langle$ ,  $\rangle \in \{\langle$ ,  $\rangle\}$, by definition of $I(bird), I(tree), I(On)$.
3. Since$\langle$ ,  $\rangle \in \{\langle$ ,  $\rangle\}$, from 1 and 2, then $M \models Under(tree, bird)$.

SOLUTION to the first part of (b).

1. $M^* \models On(bird, tree)$ iff $\langle I^*(bird), I^*(tree)\rangle \in I^*(On)$, by def. of $\models$.
2. $\langle I^*(bird), I^*(tree)\rangle \in I^*(On)$ iff $\langle$ ,  $\rangle \in \{\langle$ ,  $\rangle\}$, by definition of $I^*(bird), I^*(tree), I^*(On)$.
3. Since $\langle$ ,  $\rangle \in \{\langle$ ,  $\rangle\}$ from 1 and 2, then $M^* \models On(bird, tree)$.

SOLUTION to the second part of (b).

1. $M^* \models Under(tree, bird)$ iff $\langle I^*(tree), I^*(bird)\rangle \in I^*(Under)$, by def. of $\models$.
2. $\langle I^*(tree), I^*(bird)\rangle \in I^*(under)$ iff $\langle$ ,  $\rangle \in \{\langle$ ,  $\rangle\}$, by definition of $I^*(bird), I^*(tree), I^*(On)$.
3. Since $\langle$ ,  $\rangle \in \{\langle$ ,  $\rangle\}$, from 1 and 2, then $M^* \models Under(tree, bird)$.

SOLUTION to (c). The answer is no. First note that $I(bird) = I^*(tree)$ and that $I(tree) = I^*(bird)$. This means that I and I^* assign different meanings to the words "$bird$" and "$tree$". But despite that, parts (a) and (b) suggest that even under two different interpretations such as I and I^*, the same sentences will come out true. So, we cannot tell whether we assign meaning according to I (which would be the natural or intended interpretation) or according to I^* (which is the unnatural interpretation).

3 TELL ME THE DIFFERENCE [30 POINTS]

For each of the following models, write a formula that is true only relative to one model and false in all other models. Once you have found such a formula for each of the models, you should check that only one model makes it true and all the other models make it false. (For example, suppose you have found the formula φ which you claim to be true only in model M_2. You should check that $M_2 \models \varphi$, and in addition, you should check that $M_1 \not\models \varphi$ and $M_3 \not\models \varphi$.) When you come up with a formula, you may not use constant symbols. You may only use the propositional connectives, variables and quantifiers, together with the 2-place predicate R whose interpretation is defined relative to each model, as follows:

M_1 : $\begin{tikzcd} \bullet \arrow[r, bend left] & \star \arrow[l, bend left] \end{tikzcd}$ *with* $I_1(R) = \{\langle \bullet, \star\rangle, \langle \star, \bullet\rangle\}$

M_2 : $\begin{tikzcd} \bullet \arrow[loop above] & \star \arrow[loop above] \end{tikzcd}$ *with* $I_2(R) = \{\langle \bullet, \bullet\rangle, \langle \star, \star\rangle\}$

$$M_3: \qquad \begin{tikzcd} \bullet \arrow[r] & \star \arrow[loop above] \end{tikzcd} \qquad \textit{with } I_3(R) = \{\langle \bullet, \star \rangle, \langle \star, \star \rangle\}$$

SOLUTION. As for M_1, the formula is $\forall x \neg(R(x,x))$. We have that $M_1 \models \forall x \neg(R(x,x))$ because $\langle \bullet, \bullet \rangle \notin I_1(R)$ and $\langle \star, \star \rangle \notin I_1(R)$. However, $M_2 \not\models \forall x \neg(R(x,x))$ because e.g. $\langle \bullet, \bullet \rangle \in I_2(R)$. And also, $M_3 \not\models \forall x \neg(R(x,x))$ because $\langle \star, \star \rangle \in I_3(R)$.

As for M_2, the formula is $\forall x(R(x,x))$. We have that $M_2 \models \forall x(R(x,x))$ because $\langle \bullet, \bullet \rangle \in I_2(R)$ and $\langle \star, \star \rangle \in I_2(R)$. However, $M_1 \not\models \forall x(R(x,x))$ because e.g. $\langle \bullet, \bullet \rangle \notin I_1(R)$. And also, $M_3 \models \forall x(R(x,x))$ because $\langle \bullet, \bullet \rangle \notin I_3(R)$.

As for M_3, the formula is $\exists x R(x,x) \wedge \exists y \neg R(y,y)$. We have that $M_3 \models \exists x R(x,x) \wedge \exists y \neg R(y,y)$ because $\langle \star, \star \rangle \in I_3(R)$ but $\langle \bullet, \bullet \rangle \notin I_3(R)$. However, we have that $M_1 \not\models \exists x R(x,x) \wedge \exists y \neg R(y,y)$ because $\langle \star, \star \rangle \notin I_1(R)$ and $\langle \bullet, \bullet \rangle \notin I_1(R)$. Also, we have that $M_2 \not\models \exists x R(x,x) \wedge \exists y \neg R(y,y)$ because $\langle \star, \star \rangle \in I_2(R)$ and $\langle \bullet, \bullet \rangle \in I_2(R)$.

4 TRUTH CHECKING [15 POINTS]

Show—with painstaking precision—that $\forall x \exists y(R(x,y))$ is true in model M with $D = \{\bullet, \star\}$ and $I(R) = \{\langle \bullet, \star \rangle, \langle \star, \bullet \rangle\}$ and g the variable assignment function such that $g(x) = \bullet$ and $g(y) = \star$. You should be very precise and detailed, and in particular, you are expected to break down your proof in clearly identifiable steps and each step should appeal either to facts about the model (such as the definition of I and D) or to the base case(s) or the inductive cases of the definition of truth in a model.

SOLUTION. We unpack $M \models \forall x \exists y(R(x,y))$, as follows:

1. $\langle D, I, g \rangle \models \forall x \exists y(R(x,y))$ iff for all d [if $d \in D$, then $\langle D, I, g_{[x:=d]} \rangle \models \exists y(R(x,y))$.

   This iff-claim holds in virtue of the definition of truth in a model for universally quantified formulas.

2. for all d [if $d \in D$, then $\langle D, I, g_{[x:=d]} \rangle \models \exists y(R(x,y))$ iff

   for all d [if $d \in D$, there is a $d' \in D$ and $\langle D, I, g_{[x:=d][y:=d']} \rangle \models R(x,y)$

   This iff-claim holds in virtue of the definition of truth in a model for existentially quantified formulas.

3. for all d [if $d \in D$, there is a $d' \in D$ and $\langle D, I, g_{[x:=d][y:=d']} \rangle \models R(x,y)$ iff

   for all d [if $d \in D$, there is a $d' \in D$ and $\langle g_{[x:=d][y:=d']}(x), g_{[x:=d][y:=d']}(y) \rangle \in I(R)$

   This iff-claim is true in virtue of the definition of truth in a model for formulas containing a 2-place predicate.

4. for all d [if $d \in D$, there is a $d' \in D$ and $\langle g_{[x:=d][y:=d']}(x), g_{[x:=d][y:=d']}(y)\rangle \in I(R)$ iff for all d [if $d \in D$, there is a $d' \in D$ and $\langle g_{[x:=d]}(x), g_{[y:=d']}(y)\rangle \in I(R)$
This iff-claim holds because $g_{[x:=d][y:=d']}(x) = g_{[x:=d]}(x)$ and $g_{[x:=d][y:=d']}(y) = g_{[y:=d']}(y)$.

Here are two facts about the model to note:
(I) $\langle g_{[x:=\bullet]}(x), g_{[y:=\star]}(y)\rangle \in I(R)$ because $\langle \bullet, \star\rangle \in \{\langle \bullet, \star\rangle, \langle \star, \bullet\rangle\}$.
(II) $\langle g_{[x:=\star]}(x), g_{[y:=\bullet]}(y)\rangle \in I(R)$ because $\langle \star, \bullet\rangle \in \{\langle \bullet, \star\rangle, \langle \star, \bullet\rangle\}$.
And from (I) and (II), and the fact that $D = \{\bullet, \star\}$, it follows that
(III) for all d [if $d \in D$, there is a $d' \in D$ and $\langle g_{[x:=d]}(x), g_{[y:=d']}(y)\rangle \in I(R)$.
Finally, from (III) and the "unpacking steps" 4 through 1, we have $M \models \forall x \exists y(R(x, y))$.

5 CARTESIAN PRODUCT [15 POINTS]

Let A be a set of objects. The set $A \times A$ is called the Cartesian product and is defined as follows:

$$A \times A = \{\langle i, j\rangle | i \in A \textit{ and } j \in A\}$$

In order words, $A \times A$ is the set of all ordered pairs of elements such that each element of those pairs is an element of the set A. To illustrate, consider the set $A = \{1, 2\}$, where $A \times A = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle, \langle 2, 1\rangle\}$. Now, please do the following:

(a) Show that if $A \subseteq A^*$, then $A \times A \subseteq A^* \times A^*$.

(A and A^* are sets of objects.)

(b) Check whether $\overline{A \times A} = \overline{A} \times \overline{A}$. If the equality holds, please prove it, and if it does not, please offer a counterexample.

(Keep in mind two things. *First thing*: $\overline{A}$ is the complement of A. So, if you have a universe of three elements, say, consisting of a, b, c with $A = \{a, b\}$, then $\overline{A} = \{c\}$. In other words, $\overline{A}$ contains all the elements that are not in A, i.e. $x \in \overline{A}$ iff $x \notin A$. *Second thing:* Note that the set $\overline{A \times A}$ is obtained, first, by taking the Cartesian product $A \times A$, and second, by taking the complement of $A \times A$. Instead the set $\overline{A} \times \overline{A}$ is obtained, first, by taking the complement of A and A, and second, by taking the Cartesian product of the two sets. So, $\overline{A \times A}$ is the complement of the Cartesian product, while $\overline{A} \times \overline{A}$ is the Cartesian product of the complements.)

SOLUTION to (a). On the assumption (1) that $A \subseteq A^*$, we need to show that if $\langle i, j\rangle \in A \times A$, then $\langle i, j\rangle \in A^* \times A^*$ for an arbitrary pair $\langle i, j\rangle$. To this end, let's assume (2) that $\langle i, j\rangle \in A \times A$, and let's aim to show that $\langle i, j\rangle \in A^* \times A^*$. Here is a step-by-step proof:
1. $\langle i, j\rangle \in A \times A$, by assumption (2).

2. $\langle i, j\rangle \in A \times A$ iff $i \in A$ *and* $j \in A$ by definition of $A \times A$.
3. $i \in A$ *and* $j \in A$ by logic from 1 and 2.
4. $i \in A^*$ *and* $j \in A^*$ because of assumption (1), i.e. $A \subseteq A^*$.
5. $\langle i, j\rangle \in A^* \times A^*$ iff $i \in A^*$ *and* $j \in A^*$ by definition of $A^* \times A^*$.
6. $\langle i, j\rangle \in A^* \times A^*$, by logic from 4 and 5.

SOLUTION to (b). The claim is false. We will now construct a counterexample. Consider $A = \{1, 2\}$ and $\overline{A} = \{3\}$. We can check that $\langle 1, 3\rangle \in \overline{A \times A}$, although $\langle 1, 3\rangle \notin \overline{A} \times \overline{A}$. How so? First note that a pair $\langle i, j\rangle \in \overline{A \times A}$ iff $i \notin A$ *or* $j \notin A$. Check the footnote to see why.[1] The pair $\langle 1, 3\rangle$ satisfies this condition, because $3 \notin A$, so $\langle 1, 3\rangle \in \overline{A \times A}$. Second note that a pair $\langle i, j\rangle \in \overline{A} \times \overline{A}$ iff $i \notin A$ *and* $j \notin A$. Check the footnote to see why.[2] The pair $\langle 1, 3\rangle$ does not satisfy this condition, because $1 \in A$, so $\langle 1, 3\rangle \notin \overline{A} \times \overline{A}$. This completes the counterexample.

[1] Here are the steps:
- $\langle i, j\rangle \in \overline{A \times A}$ iff $\langle i, j\rangle \notin A \times A$ by definition of complementation.
- $\langle i, j\rangle \notin A \times A$ iff it is not the case that $[i \in A$ *and* $j \in A]$, by definition of $D \times D$.
- it is not the case that $[i \in A$ *and* $j \in A]$ iff $i \notin A$ *or* $j \notin A$, by logic.

[2] Here are the steps:
- $\langle i, j\rangle \in \overline{A} \times \overline{A}$ iff $i \in \overline{A}$ *and* $j \in \overline{A}$, by definition of $D \times D$.
- $i \in \overline{A}$ *and* $j \in \overline{A}$ iff $i \notin A$ *and* $j \notin A$.