PHIL 50 – INTRODUCTION TO LOGIC

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HOMEWORK – WEEK #9 - SOLUTIONS

1 IS IT VALID OR NOT? [50 POINTS]

(a) Construct a derivation showing that $\forall x\varphi(x) \vdash \exists x\varphi(x)$. That is, construct a derivation with uncanceled assumption $\forall x\varphi(x)$ and ending with the formula $\exists x\varphi(x)$. [*Hint*: have a look at the derivations in the derivation guide for predicate logic.]

(b) Show that $\forall x\varphi(x) \not\models \exists x\varphi(x)$ by constructing a model M consisting of a (possibly empty) domain D of objects, an interpretation function I for constant and predicate symbols, and an assignment function g for variable symbols.

(c) What do (a) and (b) show? Do they contradict soundness or completeness of predicate logic? In order to recover soundness or completeness of predicate logic, decide how the definition of model M given in part (b) should be amended.

SOLUTION to (a). There is a quick route, as follows:

$$\dfrac{\dfrac{\forall x\varphi(x)}{\varphi(x)}\ \forall E}{\exists x\varphi(x)}\ \exists I$$

There is also a much more complicated route (certainly unnecessary). We proceed in two steps. The first step is below:

$\vdash \forall x\varphi(x) \to \neg\forall x\neg\varphi(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\varphi(x)]^1}{\varphi(x)}\ \forall E \qquad \dfrac{[\forall x\neg\varphi(x)]^2}{\neg\varphi(x)}\ \forall E}{\bot}\ \to E}{\neg\forall x\neg\varphi(x)}\ \to I^2}{\forall x\varphi(x) \to \neg\forall x\neg\varphi(x)}\ \to I^1$$

And now the second step:

$\vdash \neg\forall x\neg\varphi(x) \to \exists x\varphi(x)$

$$\dfrac{\dfrac{\dfrac{[\neg\forall x\neg\varphi(x)]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg\exists x\varphi(x)]^2 \qquad \dfrac{[\varphi(x)]^3}{\exists x\varphi(x)}\,\exists I}{\bot}\,\to E}{\neg\varphi(x)}\,\to I^3}{\forall x\neg\varphi(x)}\,\forall I}{\bot}\,\to E}{\exists x\varphi(x)}\,RAA^2}{\neg\forall x\neg\varphi(x) \to \exists x\varphi(x)}\,\to I^1$$

We can now put the two earlier steps together:

$$\dfrac{\dfrac{\neg\forall x\neg\varphi(x) \to \exists x\varphi(x) \qquad \dfrac{\forall x\varphi(x) \to \neg\forall x\neg\varphi(x) \qquad [\forall x\varphi(x)]^1}{\neg\forall x\neg\varphi(x)}\,\to E}{\exists x\varphi(x)}\,\to E}{\forall x\varphi(x) \to \exists x\varphi(x)}\,\to I^1$$

SOLUTION to (b). Take model M with $D = \emptyset$, so that $M \models \forall x\varphi(x)$. To see why, recall that $M \models \forall x\varphi(x)$ iff for all d, if $d \in D$, then $\langle D, I, g_{[x:=d]}\rangle \models \varphi(x)$. Now since no d is in D because D is empty, the formula $\varphi(x)$ is true vacuously in M. By contrast, $M \not\models \exists x\varphi(x)$ because $M \models \exists x\varphi(x)$ iff there is a d such that $d \in D$ and $\langle D, I, g_{[x:=d]}\rangle \models \varphi(x)$. But there is no such d because there is no d in D, whence $M \not\models \exists x\varphi(x)$.

SOLUTION to (c). Soundness ensures that if $\forall x\varphi(x) \vdash \exists x\varphi(x)$, then $\forall x\varphi(x) \models \exists x\varphi(x)$. Part (a) and (b) together contradict soundness. One obvious way to restore soundness is to require that the domain D always be non-empty. In fact, this is how domains are defined in logic textbooks; see e.g. the definition of model M in our textbook.

2 RUTH BARCAN MARCUS [20 POINTS]

Read the *New York Times* obituary about Ruth Barcan Marcus. The weblik is posted on the course website. How does the article explain the difference between $\Box\forall x\varphi(x)$ and $\forall x\Box\varphi(x)$?

SOLUTION. The article explains the difference between $\Box\forall x\varphi(x)$ and $\forall x\Box\varphi(x)$, as follows:

> 'Necessarily, everything is physical' and 'Everything is necessarily physical.' They sound as if they mean the same, and a grammarian might say, 'What's the difference?' But in logic they're quite different.

> The difference hinges on how much of the sentence the modal word modifies. In Sentence 1, "Necessarily, everything is physical," the word "necessarily" casts a wide semantic net: it takes into account not only the real world, but also any hypothetical ones.
>
> "The comma is a giveaway," Professor Neale said. "You're saying, 'in every possible world, everything is physical.' "
>
> In Sentence 2, "Everything is necessarily physical," "necessarily" has a narrower scope: it ignores the merely possible and attends only to what actually exists. This sentence means, roughly, "Everything existing in this world has the property of being physical in every world."

3 SAUL KRIPKE [30 POINTS]

The American philosopher Saul Kripke argued against dualism in philosophy of mind using modal logic. A central question in philosophy of mind is whether mental states can be identified with (or are the same thing as) physical or brain states. In other words, is the mind the same thing as the brain or not? If mental states are the same as physical or brain states, then everything would be reducible to the physical level, and dualism would thus be false. By contrast, if mental states are not identical to physical states, then not everything would be reducible to the physical level, and dualism would thus be true.

Kripke wanted to offer an argument in defense of dualism—i.e., an argument for the claim that mental states cannot be identified with brain states. For more background on this, read the lecture notes on modal logic (and in particular, the last section). Kripke's argument rests on the claim that

$$\models (a = b) \rightarrow \Box(a = b)$$

Show that if RIGIDITY holds, the above claim holds, i.e. show that if RIGIDITY holds, the formula $(a = b) \rightarrow \Box(a = b)$ is valid. (See the lecture notes on modal logic, last section, for a statement of RIGIDITY). At some point in the proof, you will need to appeal to an *ad hoc* assumption, namely that $I_w(=)$ is the same as $I_v(=)$ for any worlds w and v.

SOLUTION. In order to prove the validity, we should assume that $M, w \models a = b$, for some arbitrary model M and world w, and then we should aim to show that $M, w \models \Box(a = b)$. So, consider an arbitrary model M and world w such that $M, w \models (a = b)$, for some constant symbols a and b. By definition, we have that

(*) $\langle I_w(a), I_w(b)\rangle \in I_w(=)$.

Since RIGIDITY holds, we have that $I_v(a)$ is the same as $I_w(a)$, for any possible world v, and also, $I_v(b)$ is the same as $I_w(b)$, again for any possible world v. So, by replacing $I_w(a)$ with $I_v(a)$ and by replacing $I_w(b)$ with $I_v(b)$, we have that

(**) for any world v, it holds $\langle I_v(a), I_v(b)\rangle \in I_v(=)$.

The above holds provided

(1) $I_v(a)$ is the same as $I_w(a)$ and $I_v(b)$ is the same as $I_w(b)$
(2) $\langle I_w(a), I_w(b)\rangle \in I_w(=)$
(3) $I_w(=)$ is the same as $I_v(=)$

Now, (1) holds because of RIGIDITY. Condition (2) holds because of (*). And (3) holds as our *ad hoc* assumption. Now, since (**) holds, by definition, $M, w \models \Box(a = b)$. Thus, $M, w \models (a = b) \rightarrow \Box(a = b)$. But M and w were arbitrary, so $\models (a = b) \rightarrow \Box(a = b)$.