## PHIL 50 – INTRODUCTION TO LOGIC – SPRING 2014

### FINAL EXAM – FRIDAY JUNE 6TH – 9:30-11:30 AM – SOLUTIONS

*The final exam consists of PART A and PART B. The former consists of ten questions which you are expected to answer as clearly and as concisely as possible. The latter part consists of five problems. The time allotted for the final exam is 2 hours, from 9:30 AM until 11:30 AM.* ***You may not consult textbooks, notes and other study materials during the exam.***

## PART A

1. Write the square of oppositions in the language of set theory.

   All A are B, that is, $A \subseteq B$

   No A is B, that is, $A \cap B = \emptyset$

   Some A are B, that is, $A \cap B \neq \emptyset$

   Not all A are B, that is, $A \not\subseteq B$

2. Translate into predicate logic the statement *All houses in Santorini are colorful*.

   $\forall x((\mathit{House(x)} \wedge \mathit{In}(x, \mathit{santorini})) \rightarrow \mathit{Colorful\ (x)})$

3. What is a model in predicate logic?

   A model consists of a domain $D$, an interpretation $I$ for constant and predicate symbols, and an assignment function $g$ for variable symbols.

4. What is the truth condition for a universally quantified formula? State the condition and give an example.

   Truth condition: $\langle D, I, g\rangle \models \forall x\varphi(x)$ iff for all $d$, if $d \in D$, then $\langle D, I, g_{[x:=d]}\rangle\varphi(x)$. Example: Consider a model such that $D = \{\star, \diamond\}$ and $I(P) = D$. Then, the formula $\forall xP(x)$ would be true in the model.

5. Why do we need a modified variable assignment of the form $g_{[x:=d]}(x)$?

   We use the modified variable assignment in the truth conditions for existentially and universally quantified formulas. Without the modified variable assignment, we would not be able to state the truth conditions for quantified formulas.

6. What restriction governs $\forall I$? What is its rationale? Please give an example.

   The restriction in the application of $\forall I$ to obtain $\forall x\varphi(x)$ is that $x$ should not occur free in any uncanceled assumption on which $\varphi(x)$ depends. The restriction that $x$ cannot occur free captures the idea that the $x$ for which we have established $\varphi(x)$ should be arbitrary. One example is Euclid's proof that for all triangles the sum of the internal angles equal two right angles. Euclid proves the claim for an arbitrary triangle and then extends the claim to all triangles. Euclid's reasoning step from an arbitrary triangle to all triangle is similar to the application of rule $\forall I$.

7. What is the transformative power of negation with respect to the quantifiers? Illustrate this with an example.

   The transformative power of negation is that when a negation moves from the inside to the outside of a quantifier (or from the outside to the inside of a quantifier), the passage of the negation changes the quantifier from existential to universal or from universal to existential. Consider, for instance, the formula $\forall x\exists y\neg P(x, y)$. This formula is equivalent to $\neg\exists x\forall y P(x, y)$.

8. How can you show that $\varphi \nvdash \psi$?

   It is enough to construct a model that makes true $\varphi$ and makes false $\psi$. By soundness, this shows that $\varphi \nvdash \psi$.

9. What is the difference between $\forall x\Box\varphi(x)$ and $\Box\forall x\varphi(x)$?

   The first formula says that for all objects (in the domain of some fixed possible world) it holds that they are $\varphi$ in all possible worlds. The second formula says that all the objects in the domain of each possible world are such that they are $\varphi$. So, in the first case we are only considering all the objects in the domain associated with one possible world. In the second case, we are consider all the objects associated with the domains of all possible worlds.

10. What does it mean to say that probability theory has an underlying logic? Give a couple of examples.

    One axiom of probability theory says that $P(\top) = 1$ with $\top$ any logical tautology. This axiom makes sense provided we have a grasp of what a logical tautology is. Also, another axiom of probability theory says that $P(\varphi \wedge \psi) = P(\varphi) + P(\psi)$, provided $\varphi$ and $\psi$ form a contradiction. But again, the notion of a contradiction is a logical notion.

## PART B

1. By constructing the appropriate derivation, show that:

   (a) $\vdash ((\varphi \to \psi) \to \psi) \to (\neg\varphi \to \psi)$

$$
\dfrac{\dfrac{\dfrac{[(\varphi \to \psi) \to \psi]^3 \qquad \dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\neg\varphi]^2}{\bot}\to E}{\psi}\bot}{\varphi \to \psi}\to I^1}{\psi}\to E}{\neg\varphi \to \psi}\to I^2}{((\varphi \to \psi) \to \psi) \to (\neg\varphi \to \psi)}\to I^3
$$

   (b) $\varphi \vee \psi, \neg\varphi \vdash \psi$

$$
\dfrac{[\varphi \vee \psi] \qquad \dfrac{\dfrac{[\neg\varphi] \quad [\varphi]^1}{\bot}\to E}{\psi}\bot \qquad \dfrac{[\psi]^1}{\psi}R}{\psi}\vee E^1
$$

   (c) $\vdash \forall x(A(x) \wedge B(x)) \to (\forall x A(x) \wedge \forall x B(x))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(A(x) \wedge B(x))]^1}{A(x) \wedge A(x)}\forall E}{A(x)}\wedge E}{\forall x(A(x))}\forall I \qquad \dfrac{\dfrac{\dfrac{[\forall x(A(x) \wedge B(x))]^1}{A(x) \wedge B(x)}\forall E}{B(x)}\wedge E}{\forall x(B(x))}\forall I}{\forall x(A(x)) \wedge \forall x(B(x))}\wedge I}{\forall x(A(x) \wedge B(x)) \to \forall x(A(x)) \wedge \forall x(B(x))}\to I^1
$$

   (d) $\vdash \forall x \exists y P(x,y) \to \forall x' \exists y' P(x', y')$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\forall x \exists y P(x,y)]^1}{\exists y P(x,y)}\forall E \qquad \dfrac{[P(x,y)]^2}{\exists y' P(x,y')}\exists I}{\exists y' P(x', y')}\exists E^2}{\forall x' \exists y' P(x', y')}\forall I}{\forall x \exists y P(x,y) \to \forall x' \exists y' P(x', y')}\to I^1
$$

(e) $\vdash \exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)$

$$\dfrac{\dfrac{\dfrac{[\exists x\forall yP(x,y]^1 \qquad \dfrac{\dfrac{[\forall y(Px,y)]^2}{P(x,y)}\,\forall E}{\exists xP(x,y)}\,\exists I}{\exists xP(x,y)}\,\exists E^2}{\forall y\exists xP(x,y)}\,\forall I}{\exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)}\,\rightarrow I^1$$

2. Show that the claim $Val(\Gamma \cup \Delta) = Val(\Gamma) \cup Val(\Delta)$ is false.

   Recall that $Val(\Gamma) = \{V|\textit{for all } \varphi\textit{, if } \varphi \in \Gamma\textit{, then } V(\varphi) = 1\ \}$. And similarly for $Val(\Delta)$.

   Consider a valuation $V$ such that $V(p) = 1$ and $V(q) = 0$, where $\Gamma = \{p\}$ and $\Delta = \{q\}$. The $V$ in question belongs to $Val(\Gamma)$, so it also belongs to $Val(\Gamma) \cup Val(\Delta)$. However, the $V$ in question does not belong to $Val(\Gamma \cup \Delta)$ because $V(q) = 0$

3. Consider a model whose domain consists of the positive natural numbers, including 0. You should picture such a model as having numbers 0, 1, 2, 3, 4, 5, etc. The numbers are ordered in the standard way. Let *Greater-than* be a two-place predicate that is interpreted as the relation $>$, or more precisely, $I(\textit{Greater-than}) = \{\langle n, m\rangle | n > m\}$. Call this model $N$. Check whether the following hold or not. Explain your answers.

   (a) $N \models \exists x\exists y\neg(x = y)$
   (b) $N \models \forall x\exists y\textit{Greater-than}(y, x)$
   (c) $N \models \exists x\forall y((\neg(x = y)) \rightarrow (\textit{Greater-than}(y, x))$
   (d) $N \models \exists x\forall y\textit{Greater-than}(x, y)$

   Part (a) says that in $N$ there are at least two objects, and that's clearly true in $N$ because there are at least two numbers in $N$.

   Part (b) says that in $N$ for every number $n$ there is a number $m$ such that $m$ is greater than $n$. That's true in $N$, because given any number $n$, a number bigger than $n$ can be found.

   Part (c) says that in $N$ there is a number $n$ such that every other number is greater than $n$. That's again true in $N$, and the number in question is zero. For it is indeed true that given any number different from zero—namely, 1, 2, 3, 4, 5, etc.—it holds that such number is greater than zero.

   Part (d) says that in $N$ there is a number which is greater than every number. This is false in $N$. For take any number $n$; now there is a number greater than $n$, namely $n + 1$, so in $N$ there is no number that is greater than any number.

4. Show that the following hold or give a counterexample:

   (a) $\forall x(P(x) \vee Q(x)) \models \forall x P(x) \vee \forall x Q(x)$

   (b) $\exists x(P(x) \vee Q(x)) \models \exists x P(x) \vee \exists x Q(x)$

   (c) $\exists x P(x) \wedge \exists x Q(x) \models \exists x(P(x) \wedge Q(x))$

   Counterexample to (a). Let $D = \{\star, \diamond\}$ and let $I(P) = \{\star\}$ and $I(Q) = \{\diamond\}$. Now, in the model under consideration every object $d \in D$ is such that $d \in I(P)$ or $d \in I(Q)$, so the model makes true $\forall x(P(x) \vee Q(x))$. However, the model does not make true $\forall x P(x) \vee \forall x Q(x)$. Why? It is not the case that every object $d$ is in $I(P)$, nor is it the case that every object $d$ is in $I(Q)$, whence $\forall x P(x) \vee \forall x Q(x)$ is false in our model.

   Part (b) holds. Let's first show that $\exists x(P(x) \vee Q(x)) \vdash \exists x P(x) \vee \exists x Q(x)$, as follows:

$$\dfrac{\exists x(P(x) \vee Q(x)) \qquad \dfrac{[P(x) \vee Q(x)]^2 \qquad \dfrac{\dfrac{[P(x)]^1}{\exists x P(x)}\,\exists I}{\exists x P(x) \vee \exists x Q(x)}\,\vee I \qquad \dfrac{\dfrac{[Q(x)]^1}{\exists x Q(x)}\,\exists I}{\exists x P(x) \vee \exists x Q(x)}\,\vee I}{\exists x P(x) \vee \exists x Q(x)}\,\vee E^1}{\exists x P(x) \vee \exists x Q(x)}\,\exists E^2$$

   By soundness, it follows that $\exists x(P(x) \vee Q(x)) \models \exists x P(x) \vee \exists x Q(x)$. There is also a semantic proof, which is a bit cumbersome but which reflects the structure of the syntactic derivation, as follows.

   Suppose a model $M$ is such that $M \models \exists x(P(x) \vee Q(x))$. This means that there is a $d \in D$ such that $d \in I(P)$ or $d \in I(Q)$. We have two alternatives here.

   First, let's suppose that $d$ is in $I(P)$. This would mean that there is a $d \in D$ such that $d \in I(P)$, and consequently, there is a $d \in D$ such that $d \in I(P)$ OR there is a $d \in D$ such that $d \in I(Q)$ (We added the OR-part by a reasoning step that parallels disjunction introduction).

   And now the second alternative. Second, let's suppose that $d$ is in $I(Q)$. This would mean that there is a $d \in D$ such that $d \in I(Q)$, and consequently, there is a $d \in D$ such that $d \in I(P)$ OR there is a $d \in D$ such that $d \in I(Q)$ (We added the OR-part by a reasoning step that parallels disjunction introduction).

   All in all, no matter which alternative we pick, we arrive at the conclusion that there is a $d \in D$ such that $d \in I(P)$ OR there is a $d \in D$ such that $d \in I(Q)$. This means, that there is a $d \in D$ such that $d \in I(P)$ OR there is a $d \in D$ such that $d \in I(Q)$, and the latter means that our model $M$ is such that $M \models \exists x P(x) \vee \exists x Q(x)$. Since our model was arbitrary, the claim in (b) has been established.

Counter example to (c). Let $D = \{\star, \diamond\}$ and let $I(P) = \{\star\}$ and $I(Q) = \{\diamond\}$. Now, in the model under consideration there is an object $d \in D$ such that $d \in I(P)$ and there is an object $d \in D$ such that $d \in I(Q)$. So the model makes true $\exists x P(x) \wedge \exists x Q(x)$. However, the model does not make true $\exists x(P(x) \wedge Q(x))$. Why? There is no object $d \in D$ such that $d \in I(P)$ and $d \in I(Q)$ because $I(P) \cap I(Q) = \emptyset$.

5. Show with a semantic argument that $\forall x \Box P(x) \to \Box \forall x P(x)$ is valid provided $D_w = D_v$ for all possible worlds $w$ and $v$.

   Suppose $M, w \models \forall x \Box P(x)$. This means that

   (1) for all $d$, if $d \in D_w$, then for all possible worlds $u$, it holds that $d \in I_u(P)$.

   We want to show that $M, w \models \Box \forall x P(x)$. That is, we want to show that

   (2) for all possible worlds $u$, if $d \in D_u$, then $d \in I_u(P)$.

   Since (2) is a universally quantified if-then claim, let's assume

   (3) $d \in D_{u^*}$, with $u^*$ some arbitrary possible world.

   Now, since $D_w = D_v$ for any $w$ and $v$, we also have that $D_{u^*} = D_w$, so from $d \in D_{u^*}$, it follows that $d \in D_w$. By (1) above, it follows that for all possible worlds $u$, it holds that $d \in I_u(P)$. In particular, it holds that $d \in I_{u^*}(P)$.

   Since we have assumed that (3) $d \in D_{u^*}$ and we concluded that $d \in I_{u^*}(p)$, we can say that if $d \in D_{u^*}$, then $d \in I_{u^*}(P)$. Since $u^*$ was arbitrary, we can say that for all possible world $u$, if $d \in D_u$, then $d \in I_u(P)$. This is (2) which we aimed to show. Hence, $M, w \models \Box \forall x P(x)$.

   Finally, $M$ and $w$ we started with were arbitrary, so $\forall x \Box P(x) \to \Box \forall x P(x)$ is valid.