PHIL 50 – INTRODUCTION TO LOGIC

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SAMPLE DERIVATIONS IN PROPOSITIONAL LOGIC

1 DERIVATIONS USING THE RULES FOR $\wedge$ AND THE RULES FOR $\to$

$\vdash (\varphi \wedge \psi) \to (\psi \wedge \varphi)$

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\wedge E \quad \dfrac{[\varphi \wedge \psi]^1}{\varphi}\wedge E}{\psi \wedge \varphi}\wedge I}{(\varphi \wedge \psi) \to (\psi \wedge \varphi)} \to I^1$$

$\vdash \varphi \to (\psi \to \varphi)$

$$\dfrac{\dfrac{\dfrac{[\psi]^1}{[\varphi]^2}}{\psi \to \varphi} \to I^1}{\varphi \to (\psi \to \varphi)} \to I^2$$

$\vdash (\varphi \to \psi) \to ((\psi \to \sigma) \to (\varphi \to \sigma))$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\psi \to \sigma]^2 \quad \dfrac{[\varphi]^1 \quad [\varphi \to \psi]^3}{\psi}}{\sigma} \to E}{\varphi \to \sigma} \to I^1}{(\psi \to \sigma) \to (\varphi \to \sigma)} \to I^2}{(\varphi \to \psi) \to ((\psi \to \sigma) \to (\varphi \to \sigma))} \to I^3$$

$\vdash ((\varphi \wedge \psi) \to \sigma) \to (\varphi \to (\psi \to \sigma))$

$$\dfrac{\dfrac{\dfrac{\dfrac{[(\varphi \wedge \psi) \to \sigma]^3 \qquad \dfrac{[\varphi]^2 \quad [\psi]^1}{\varphi \wedge \psi}\wedge I}{\sigma}\to E}{\psi \to \sigma}\to I^1}{\varphi \to (\psi \to \sigma)}\to I^2}{((\varphi \wedge \psi) \to \sigma) \to (\varphi \to (\psi \to \sigma))}\to I^3$$

STRATEGY 1: Whenever you are trying to construct a derivation of a formula of the form $\varphi \to \psi$, the most natural thing to do is to assume φ, and then attempt to derive ψ. Once you have derived ψ from the assumption φ, you can finally derive $\varphi \to \psi$ by applying $\to I$ which also allows you to cancel the initial assumption φ.

This strategy applies at any stage of the derivation process. You might need to derive a formula of the form $\varphi \to \psi$ as the very last formula of your derivation, or you might need to derive a formula of the form $\varphi \to \psi$ at the beginning or in the middle of your derivation. In either case, STRATEGY 1 applies.

STRATEGY 2: Often it is useful to work backwards. Ask yourself, which rule will allow me to derive the formula I need to derive? If the formula is of the form $\varphi \to \psi$, the rule to use is $\to I$; see STRATEGY 1. If the formula is a conjunction of the form $\varphi \wedge \psi$, then you should try to derive each conjunct independently, and then apply $\wedge I$ so that you can derive $\varphi \wedge \psi$.

2 DERIVATIONS USING THE RULES FOR $\wedge$ AND THE RULES FOR $\to$ AND INVOLVING FORMULAS WITH $\neg$

$\vdash \varphi \to (\neg\varphi \to \psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\neg\varphi]^2}{\bot}\to E}{\psi}\bot}{\neg\varphi \to \psi}\to I^2}{\varphi \to (\neg\varphi \to \psi)}\to I^1$$

$\vdash \varphi \to \neg\neg\varphi$

$$\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\neg\varphi]^2}{\bot} \to E}{\neg\neg\varphi} \to I^2}{\varphi \to \neg\neg\varphi} \to I^1$$

$\vdash \neg(\varphi \wedge \neg\varphi)$

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \neg\varphi]^1}{\varphi} \wedge E \quad \dfrac{[\varphi \wedge \neg\varphi]^1}{\neg\varphi} \wedge E}{\bot} \to E}{\neg(\varphi \wedge \neg\varphi)} \to I^1$$

$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\psi]^3 \quad \dfrac{[\varphi \to \psi]^1 \quad [\varphi]^2}{\psi} \to E}{\bot} \to E}{\neg\varphi} \to I^2}{\neg\psi \to \neg\varphi} \to I^3}{(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)} \to I^1$$

NOTATIONAL CONVENTION: Do not forget our notational convention which says that formulas of the form $\neg\varphi$ are abbreviations of formulas of the form $\varphi \to \bot$. You should read the above derivations which contain formulas of the form $\neg\varphi$ while having in mind our notational convention.

STRATEGY 3: Whenever you want to derive a negated formula of the form $\neg\varphi$, try to assume φ and then derive $\bot$. By applying $\to I$, you'll then be able to derive $\varphi \to \bot$ and cancel the assumption φ. This is not much different from STRATEGY 1, although here you should keep in mind that $\varphi \to \bot$ is—by our notation convention—the same as $\neg\varphi$.

3 DERIVATIONS USING—IN ADDITION—THE RULES FOR $\bot$ AND RAA

$\vdash \neg\neg\varphi \to \varphi$

$$\dfrac{\dfrac{\dfrac{[\neg\varphi]^1 \quad [\neg\neg\varphi]^2}{\bot} \to E}{\varphi} RAA^1}{\neg\neg\varphi \to \varphi} \to I^2$$

$\vdash (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\psi]^1 \quad [\neg\psi \to \neg\varphi]^3}{\neg\varphi}\to E \qquad [\varphi]^2}{\bot}\to E}{\psi}\ RAA^1}{\varphi \to \psi}\to I^2}{(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)}\to I^3$$

4 DERIVATION USING—IN ADDITION—THE RULES FOR $\vee$

$\vdash (\varphi \to \psi) \to (\varphi \to (\psi \vee \sigma))$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi \to \psi]^2}{\psi}\to E}{\psi \vee \sigma}\vee I}{\varphi \to (\psi \vee \sigma)}\to I^1}{(\varphi \to \psi) \to (\varphi \to (\psi \vee \sigma))}\to I^2$$

$\vdash \varphi \vee \neg\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi \vee \neg\varphi}\vee I \qquad [\neg(\varphi \vee \neg\varphi)]^2}{\bot}\to E}{\neg\varphi}\to I^1}{\varphi \vee \neg\varphi}\vee I \qquad [\neg(\varphi \vee \neg\varphi)]^2}{\bot}\to E}{\varphi \vee \neg\varphi}\ RAA^2$$

$\vdash \psi \to ((\varphi \vee \sigma) \to ((\varphi \wedge \psi) \vee \sigma))$

$$\dfrac{\dfrac{\dfrac{[\varphi \vee \sigma]^3 \qquad \dfrac{\dfrac{[\varphi]^1 \quad [\psi]^2}{\varphi \wedge \psi}\wedge I}{(\varphi \wedge \psi) \vee \sigma}\vee I \qquad \dfrac{[\sigma]^1}{(\varphi \wedge \psi) \vee \sigma}\vee I}{(\varphi \wedge \psi) \vee \sigma}\vee E^1}{(\varphi \vee \sigma) \to ((\varphi \wedge \psi) \vee \sigma)}\to I^3}{\psi \to ((\varphi \vee \sigma) \to ((\varphi \wedge \psi) \vee \sigma))}\to I^2$$

$\vdash ((\varphi \vee \sigma) \wedge \psi) \to ((\varphi \wedge \psi) \vee \sigma)$

$$
\begin{prooftree}
\AxiomC{$[(\varphi \vee \sigma) \wedge \psi]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\varphi \vee \sigma$}
\AxiomC{$[\varphi]^2$}
\AxiomC{$[(\varphi \vee \sigma) \wedge \psi]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{ψ}
\RightLabel{$\wedge I$}
\BinaryInfC{$\varphi \wedge \psi$}
\RightLabel{$\vee I$}
\UnaryInfC{$(\varphi \wedge \psi) \vee \sigma$}
\AxiomC{$[\sigma]^2$}
\RightLabel{$\vee I$}
\UnaryInfC{$(\varphi \wedge \psi) \vee \sigma$}
\RightLabel{$\vee E^2$}
\TrinaryInfC{$(\varphi \wedge \psi) \vee \sigma$}
\RightLabel{$\rightarrow I^1$}
\UnaryInfC{$((\varphi \vee \sigma) \wedge \psi) \rightarrow ((\varphi \wedge \psi) \vee \sigma)$}
\end{prooftree}
$$

$\vdash ((\varphi \wedge \psi) \vee \sigma) \rightarrow (\varphi \vee \sigma) \wedge (\psi \vee \sigma)$

$$
\begin{prooftree}
\AxiomC{$[((\varphi \wedge \psi) \vee \sigma)]^3$}
\AxiomC{$[\varphi \wedge \psi]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{φ}
\RightLabel{$\vee I$}
\UnaryInfC{$\varphi \vee \sigma$}
\AxiomC{$[\sigma]^1$}
\RightLabel{$\vee I$}
\UnaryInfC{$\varphi \vee \sigma$}
\RightLabel{$\vee E^1$}
\TrinaryInfC{$\varphi \vee \sigma$}
\AxiomC{$[(\varphi \wedge \psi) \vee \sigma]^3$}
\AxiomC{$[\varphi \wedge \psi]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{ψ}
\RightLabel{$\vee I$}
\UnaryInfC{$\psi \vee \sigma$}
\AxiomC{$[\sigma]^2$}
\RightLabel{$\vee I$}
\UnaryInfC{$\psi \vee \sigma$}
\RightLabel{$\vee E^2$}
\TrinaryInfC{$\psi \vee \sigma$}
\BinaryInfC{$(\varphi \vee \sigma) \wedge (\psi \vee \sigma)$}
\RightLabel{$\rightarrow I^3$}
\UnaryInfC{$(\varphi \wedge \psi) \vee \sigma) \rightarrow (\varphi \vee \sigma) \wedge (\psi \vee \sigma)$}
\end{prooftree}
$$