MATH ON TRIAL – PROBABILITY

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1. THE PHILOSOPHY OF PROBABILITY

What do we mean when we say *there is x percent chance/probability that...*?

There are different interpretations of probability:
naive or classical (but see Bertrand's paradox)
frequency
epistemic
model-based

QUESTION: Which interpretation is the most fitting for trial proceedings?

2. THE MATHEMATICS OF PROBABILITY

P is a probability function provided:

(normality) $0 \leq P(A) \leq 1$, for any proposition A;

(certaintity) $P(\top) = 1$, with $\top$ any tautology; and

(additivity) $P(A \vee B) = P(A) + P(B)$, with A, B inconsistent propositions.

The *conditional probability* of A given some other proposition B is defined as follows:

(conditional probability) $P(A|B) = \frac{P(A \wedge B)}{P(B)}$.

Simple corollaries follow, such as:

(negation) $P(\neg A) = 1 - P(A)$;

(product) $P(A \wedge B) = P(A|B)P(B)$;

(product*) $P(A \wedge B) = P(A)P(B)$ if $P(A|B) = P(A)$;

Notation: The symbol '$\neg$' stands for negation, '$\wedge$' for conjunction, and '$\vee$' for disjunction.

3. BAYES' THEOREM

Recall that confusing $P(A|B)$ and $P(B|A)$ is known as the *inversion fallacy* or *prosecutor fallacy*. Bayes' theorem shows how the two probabilities are related, as follows:

$$P(A|B) = \frac{P(B|A)}{P(B)}P(A) = \frac{P(B|A)}{P(B|A)P(A) + P(B|\neg A)P(\neg A)}P(A).$$

Bayes' theorem allows us to calculate the *conditional* probability of A given B from:
(i) the probability $P(A)$ regardless of B;
(ii) the probability $P(B)$, where $P(B) = P(B|A)P(A) + P(B|\neg A)P(\neg A)$;
(iii) the *likelihood* $P(B|A)$, i.e. the probability of B given A.

4. Taxi cabs and base rates

Imagine that there are two taxi companies, Green Cabs Inc. and Blue Cabs Inc., whose vehicles are respectively painted green and blue. There are no other taxi companies around. On a misty day a cab hits and injures a passerby, but it drives off. A witness reports that it was a blue cab. The witness is right only 80 percent of the time. This means that his *reliability* equals 0.8 in the sense that he gets the color right 80 percent of the time. Given the witness report, what is the probability that the taxi cab involved in the accident was in fact blue?

5. Bayes' theorem—the odds formulation

Another formulation of Bayes' theorem, which makes calculations easier, is in terms of odds:

$$\frac{P(A|B)}{P(\neg A|B)} = \frac{P(B|A)}{P(B|\neg A)} \times \frac{P(A)}{P(\neg A)}.$$

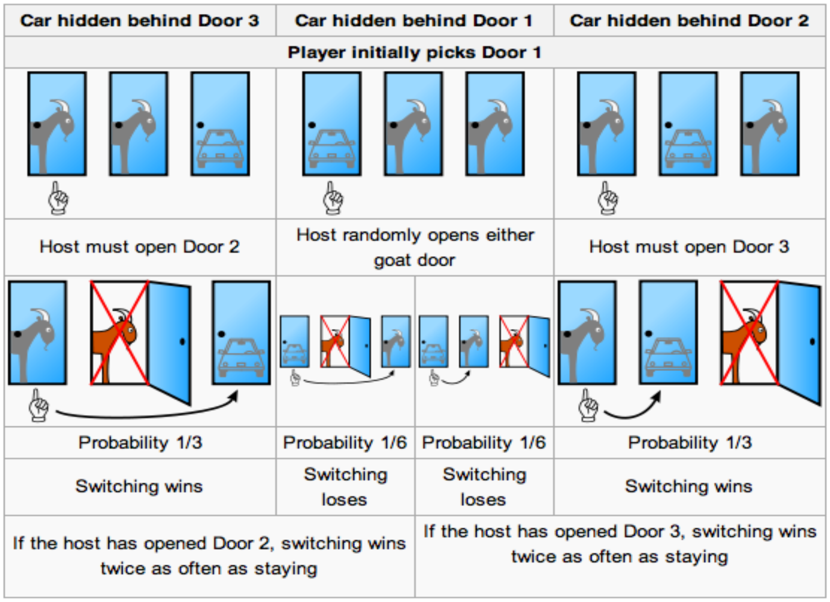
In other words,

$$\textit{posterior odds} = \textit{likelihood ratio} \times \textit{base rate odds}.$$

The posterior probability $P(A|B)$ is usually given by $\frac{PO}{1+PO}$, were PO are the posterior odds.

6. Monty Hall problem—does switching increase your chances of finding the prize?

7. SWITCHING IS BETTER: FREQUENTISTIC EXPLANATION



8. SWITCHING IS BETTER: BAYESIAN REASONING

Run Bayes' theorem; you'll see that switching increases your probability of finding the prize. How so?

Consider the following variation of the Monty Hall problem. After you have picked, say, door 1, the host opens another door *at random* (either door 2 or door 3) and behind it there is no prize. Would switching increase your chances of finding the prize? How does this scenario differ from the standard Monty Hall problem?