2.6 Valid Consequence and Consistency

We now define the general notion of valid consequence for propositional logic. It is a more precise version of the notion of a valid argument that we introduced on page 2-4.

The notion runs over all possible valuations, and as we will see in a moment, we can use truth tables to check given inferences for validity. (In what follows, k can be any number. If it is $k = 0$, then there are no premises.)

Definition 2.13 (Valid consequence) The inference from a finite set of premises

$$\varphi_1, \dots, \varphi_k$$

to a conclusion ψ is a *valid consequence*, something for which we write

$$\varphi_1, \dots, \varphi_k \models \psi,$$

if each valuation V with $V(\varphi_1) = \dots = V(\varphi_k) = 1$ also has $V(\psi) = 1$.

Definition 2.14 (Logical equivalence) If $\varphi \models \psi$ and $\psi \models \varphi$ we say that φ and ψ are *logically equivalent*.

Here it is useful to recall a warning that was already stated above. Do not confuse valid consequence with truth of formulas in a given situation: validity quantifies over truth in many situations, but it has no specific claim about truth or falsity of the premises and conclusions in the situation. Indeed, validity rules out surprisingly little in this respect: of all the possible truth/falsity combinations that might occur for premises and conclusion, it only rules out one case: viz. that all φ_i get value 1, while ψ gets value 0.

Another point from Section 2.2 that is worth repeating here concerns the role of propositional inference in conversation and argumentation. Valid inference does not just help establish truth, but it can also achieve a refutation of claims: when the conclusion of a valid consequence is false, at least one of the premises must be false. But logic does not tell us in general which one: some further investigation may be required to find the culprit(s). It has been said by philosophers that this refutational use of logic may be the most important one, since it is the basis of *learning*, where we constantly have to give up current beliefs when they contradict new facts.

Here is a simple example of how truth tables can check for validity:

Example 2.15 (Modus Tollens) The simplest case of refutation depends on the rule of *modus tollens*:

$$\varphi \to \psi, \neg\psi \models \neg\varphi.$$

Below you see the complete truth table demonstrating its validity:

φ	ψ	$\varphi \to \psi$	$\neg\psi$	$\neg\varphi$	
1	1	**1**	**0**	**0**	
1	0	**0**	**1**	**0**	
0	1	**1**	**0**	**1**	
0	0	**1**	**1**	**1**	!!

(2.22)

Of the four possible relevant situations here, only one satisfies both premises (the valuation on the fourth line), and we can check that there, indeed, the conclusion is true as well. Thus, the inference is valid.

By contrast, when an inference is invalid, there is at least one valuation (i.e., a line in the truth table) where its premises are all true, and the conclusion false. Such situations are called *counter-examples*. The preceding table also gives us a counter-example for the earlier invalid consequence

from $\varphi \to \psi, \neg\varphi$ to $\neg\psi$

namely, the valuation on the third line where $\varphi \to \psi$ and $\neg\varphi$ are true but $\neg\psi$ is false.

Please note that invalidity does not say that *all* valuations making the premises true make the conclusion false. The latter would express a valid consequence again, this time, the 'refutation' of ψ (since $\neg\varphi$ is true iff φ is false):

$$\varphi_1, \ldots, \varphi_k \models \neg\psi \tag{2.23}$$

Satisfiability Finally, here is another important logical notion that gives another perspective on the same issues:

Definition 2.16 (Satisfiable) A set of formulas X (say, $\varphi_1, \ldots, \varphi_k$) is *satisfiable* if there is a valuation that makes all formulas in X true.

There is a close connection between *satisfiability* and *consistency*.

Satisfiable versus Consistent A set of formulas that does not lead to a contradiction is called a *consistent* formula set. Here 'leading to a contradiction' refers to proof rules, so this is a definition in terms of proof theory. But it is really the other side of the same coin, for a set of formulas is consistent iff the set is satisfiable. Satisfiability gives the semantic perspective on consistency.

Instead of 'not consistent' we also say *inconsistent*, which says that there is no valuation where all formulas in the set are true simultaneously.

Satisfiability (consistency) is not the same as truth: it does not say that all formulas in X are actually true, but that they could be true in some situation. This suffices for many purposes. In conversation, we often cannot check directly if what people tell us is true (think of their accounts of their holiday adventures, or the brilliance of their kids), but we often believe them as long as what they say is consistent. Also, as we noted in Chapter 1, a lawyer does not have to prove that her client is innocent, she just has to show that it is consistent with the given evidence that he is innocent.

We can test for consistency in a truth table again, looking for a line making all relevant formulas true. This is like our earlier computations, and indeed, validity and consistency are related. For instance, it follows directly from our definitions that

$$\varphi \models \psi \text{ if and only if } \{\varphi, \neg\psi\} \text{ is not consistent.} \tag{2.24}$$

Tautologies Now we look briefly at the 'laws' of our system:

Definition 2.17 (Tautology) A formula ψ that gets the value 1 in every valuation is called a *tautology*. The notation for tautologies is $\models \psi$.

Many tautologies are well-known as general laws of propositional logic. They can be used to infer quick conclusions or simplify given assertions. Here are some useful tautologies:

$$\begin{array}{ll} \text{Double Negation} & \neg\neg\varphi \leftrightarrow \varphi \\ \text{De Morgan laws} & \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi) \\ & \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi) \\ \text{Distribution laws} & (\varphi \wedge (\psi \vee \chi)) \leftrightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)) \\ & (\varphi \vee (\psi \wedge \chi)) \leftrightarrow ((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \end{array} \tag{2.25}$$

Check for yourself that they all get values 1 on all lines of their truth tables.

Tautologies are a special zero-premise case of valid consequences, but via a little trick, they encode all valid consequences. In fact, every valid consequence corresponds to a tautology, for it is easy to see that:

$$\varphi_1, \ldots, \varphi_k \models \psi \text{ if and only if } (\varphi_1 \wedge \ldots \wedge \varphi_k) \to \psi \text{ is a tautology} \tag{2.26}$$

Exercise 2.18 Using a truth table, determine if the two formulas

$$\neg p \to (q \vee r), \neg q$$

together logically imply

(1) $p \wedge r$.

(2) $p \vee r$.

Display the complete truth table, and use it to justify your answers to (1) and (2).

Exercise 2.19

Show using a truth table that:

- the inference from $p \to (q \wedge r)$, $\neg q$ to $\neg p$ is valid and
- the inference from $p \to (q \vee r)$, $\neg q$ to $\neg p$ is not valid.

Exercise 2.20 Check if the following are valid consequences:

(1) $\neg(q \wedge r), q \models \neg r$

(2) $\neg p \vee \neg q \vee r, q \vee r, p \models r$.

Exercise 2.21 Give truth tables for the following formulas:

(1) $(p \vee q) \vee \neg(p \vee (q \wedge r))$

(2) $\neg((\neg p \vee \neg(q \wedge r)) \vee (p \wedge r))$

(3) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

(4) $(p \leftrightarrow (q \to r)) \leftrightarrow ((p \leftrightarrow q) \to r)$

(5) $((p \leftrightarrow q) \wedge (\neg q \to r)) \leftrightarrow (\neg(p \leftrightarrow r) \to q)$

Exercise 2.22 Which of the following pairs are *logically equivalent*? Confirm your answer using truth tables:

(1) $\varphi \to \psi$ and $\psi \to \varphi$

(2) $\varphi \to \psi$ and $\neg\psi \to \neg\varphi$

(3) $\neg(\varphi \to \psi)$ and $\varphi \vee \neg\psi$

(4) $\neg(\varphi \to \psi)$ and $\varphi \wedge \neg\psi$

(5) $\neg(\varphi \leftrightarrow \psi)$ and $\neg\varphi \leftrightarrow \neg\psi$

(6) $\neg(\varphi \leftrightarrow \psi)$ and $\neg\varphi \leftrightarrow \psi$

(7) $(\varphi \wedge \psi) \leftrightarrow (\varphi \vee \psi)$ and $\varphi \leftrightarrow \psi$