Euclid *Archimedes* *Gerhard Gentzen*

PHIL 50 - Introduction to Logic

Marcello Di Bello, Stanford University, Spring 2014

Week 3 — Monday Class - Derivations in Propositional Logic

The Semantic and the Syntactic Perspective

- Logical Consequence

$$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$$

iff

all valuations *V*'s that make $\phi_1, \phi_2, \ldots, \phi_k$ true also make ψ true

- Derivability

$$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$$

iff

there is a derivation whose assumptions are $\phi_1, \phi_2, \ldots, \phi_k$ and whose conclusion is ψ

Reminder — What's a Valuation?

- *V* assigns a truth value 1 or 0 to **all atomic formulas**.

- *V* is **extended to all formulas** according to the **meaning of the connectives** as defined by the truth tables. So, *V* assigns a truth value 0 or 1 to **all formulas**.

What Does a Derivation Look Like?

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi \vee \neg\varphi}\,\vee I \qquad [\neg(\varphi \vee \neg\varphi)]^2}{\bot}\,\to E}{\neg\varphi}\,\to I^1}{\varphi \vee \neg\varphi}\,\vee I \qquad [\neg(\varphi \vee \neg\varphi)]^2}{\bot}\,\to E}{\varphi \vee \neg\varphi}\,RAA^2$$

Derivations Rules for Today

$$\frac{\phi}{\phi}\,\mathbf{R}$$

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\,\wedge\mathbf{I}$$

$$\frac{\phi \wedge \psi}{\phi}\,\wedge\mathbf{E}$$

$$\frac{\phi \wedge \psi}{\psi}\,\wedge\mathbf{E}$$

$$\frac{\begin{matrix}[\phi]^{i} \\ \vdots \\ \psi\end{matrix}}{\phi \to \psi}\,\to\mathbf{I}^{i}$$

$$\frac{\phi \quad \phi \to \psi}{\psi}\,\to\mathbf{E}$$

Let’s Begin with an Argument in Natural Language

David Hume in the *Dialogues Concerning Natural Religion* (1779)

Nothing is demonstrable unless the contrary implies a contradiction. Nothing that is distinctively conceivable implies a contradiction. Whatever we conceive as existent, we can also conceive as non-existent. There is no being, therefore, whose non-existence implies a contradiction. Consequently, there is no being whose existence is demonstrable.

Looking More Closely

(1) Nothing is demonstrable unless the contrary implies a contradiction. (2) Nothing that is distinctively conceivable implies a contradiction. (3) Whatever we conceive as existent, we can also conceive as non-existent. (4) There is no being, therefore, whose non-existence implies a contradiction (5) Consequently, there is no being whose existence is demonstrable.

Is the argument valid?

(2) (3)

———————

(4) (1)

———————————

(5)

Derivations will have this tree-like structure

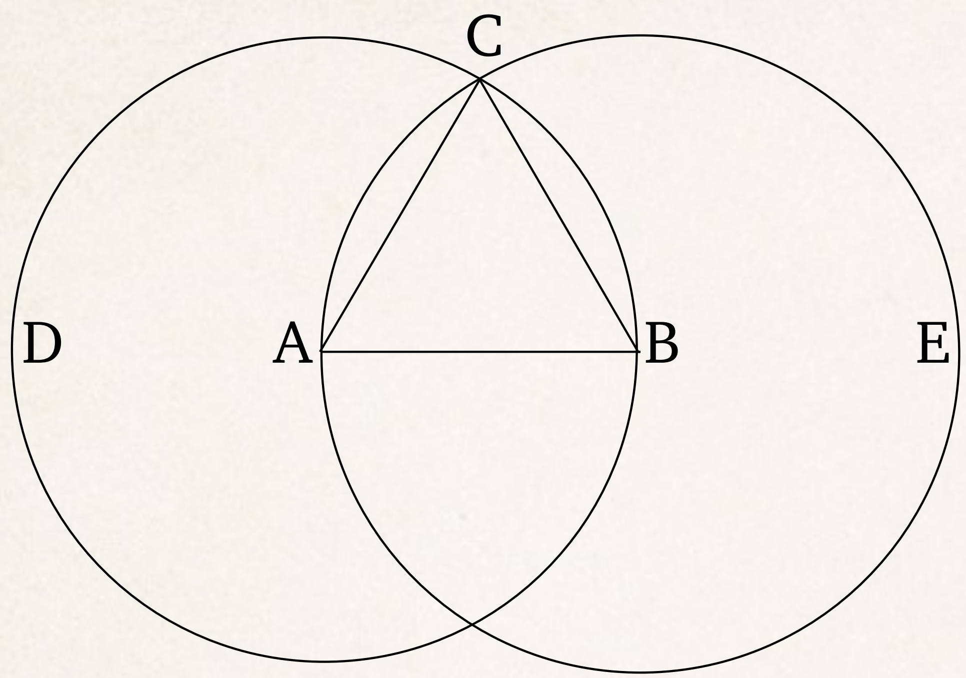
And Now an Example from Geometry

Euclid's *Elements*

(circa 300 BC)

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Def. 15: A circle is a plane figure contained by a single line called a circumference, such that all of the straight-lines radiating towards the circumference from one point amongst those lying inside the figure are equal to one another.

C.N. 1: Things equal to the same thing are also equal to one another.

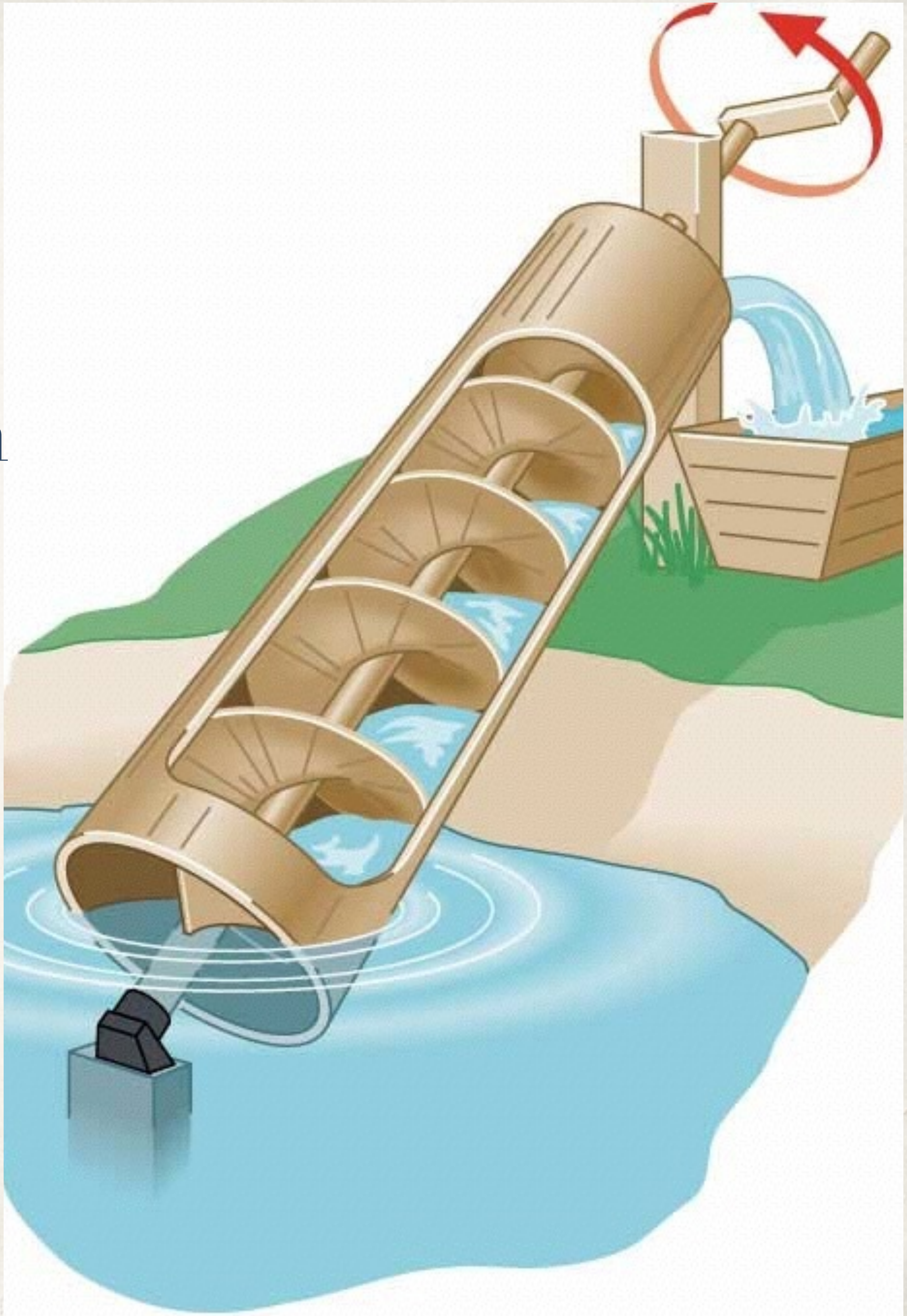
Def. 15

AC=AB BC=BA C.N. 1

CA=BC

CA=AB=BC

A Hero of the Euclidean Method: Archimedes (287-212 BC)



Too Much Abstraction May Be Fatal…

As Archimedes was drawing diagrams with mind and eyes fixed on the ground, a soldier who had broken into the house in quest of loot with sword drawn over his head asked him who he was. Too much absorbed in tracking down his objective, Archimedes could not give his name but said, protecting the dust with his hands, “I beg you, don’t disturb this,” and was slaughtered as neglectful of the victor’s command; with his blood he confused the lines of his art.

(Valerius Maximus, *Memorable Doings and Sayings*)

Archimedes' Death by the hands of Roman soldier, Luca Giordano (1632-1705)

Abstractions versus Practicalities

The death of Archimedes by the hands of a Roman soldier is symbolical of a world-change of the first magnitude: the Greeks, with their love of abstract science, were superseded in the leadership of the European world by the practical Romans. …

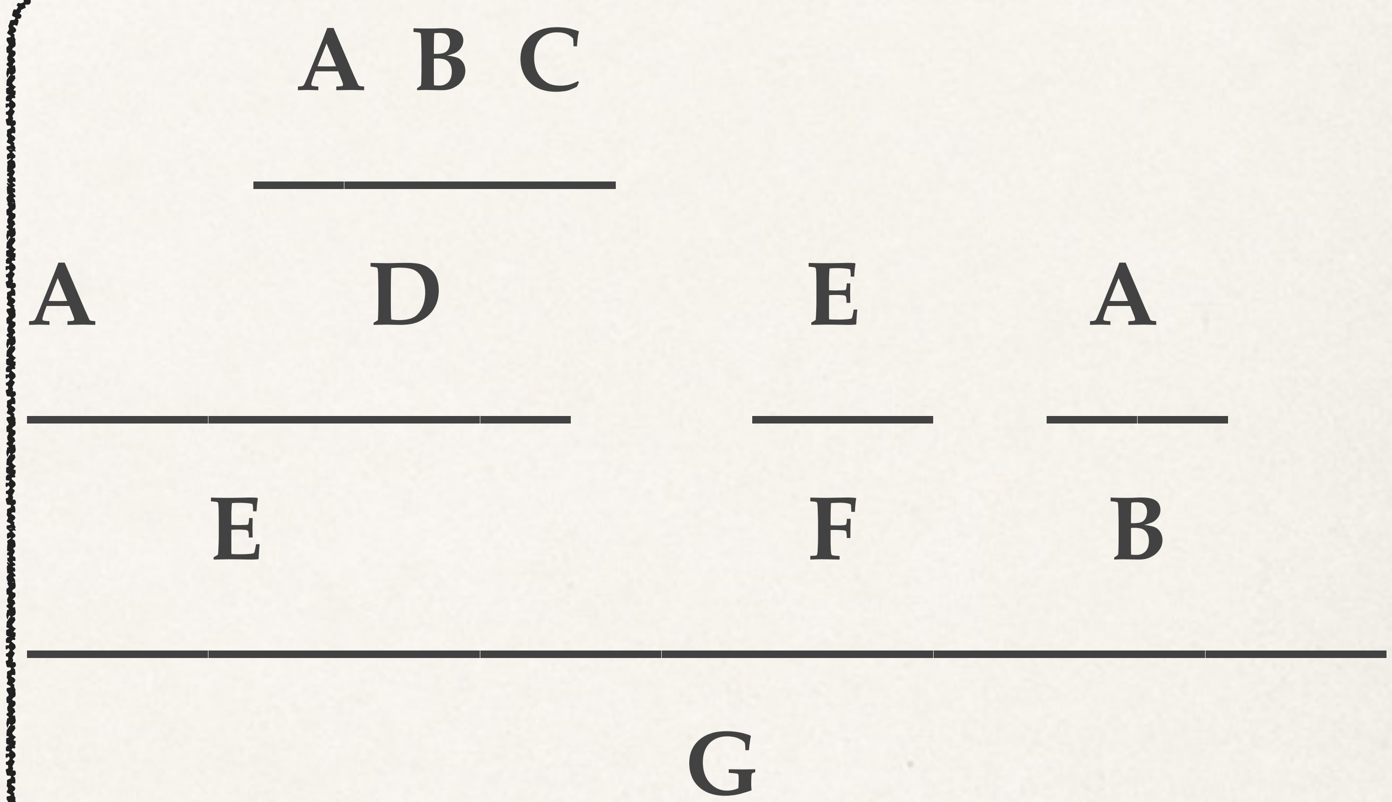
The Romans were a great race, but they were cursed with the sterility which waits upon practicality. They did not improve upon the knowledge of their forefathers, and all their advances were confined to the minor technical details of engineering.

They were not dreamers enough to arrive at new points of view, which could give a more fundamental control over the forces of nature. No Roman lost his life because he was absorbed in the contemplation of a mathematical diagram.

Whitehead, *An Introduction to Mathematics*, 1911.

Derivations as Tree-like Structures

Suppose that **each step in the derivation is valid** and that **A is true** and **G is false**. *Which letters will have to be true or false?*



A and B must be true. E, D, and C must be false.

By arranging statements in tree-like forms we can easily see their inferential relationships.

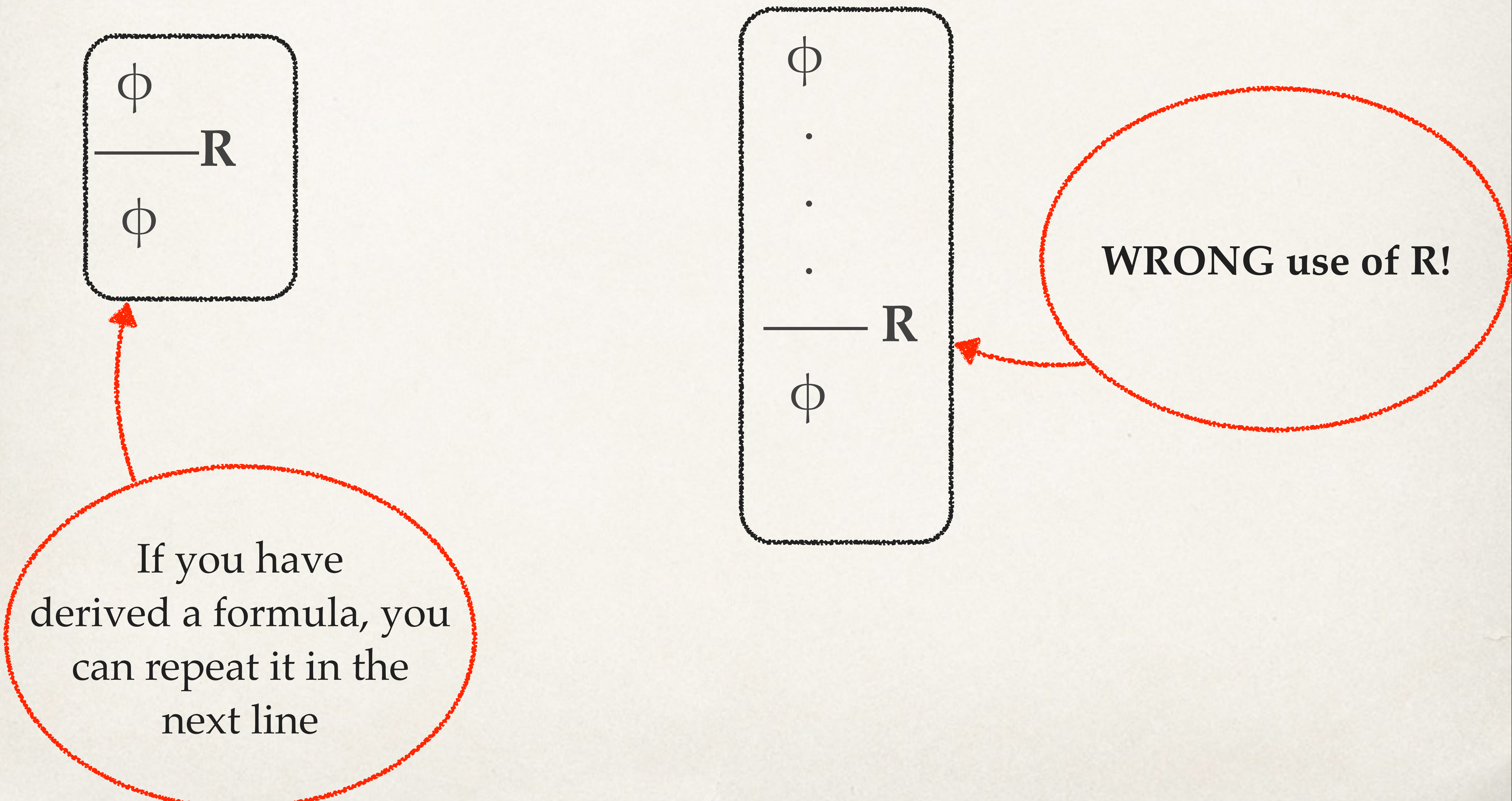
Gentzen's *Investigations into Logical Deduction (1935)*

The investigations that follow … comprise the ***types of inferences that are continually used in all parts of mathematics****. What remains to be added to these are axioms and forms of inference that may be considered as being proper of particular branches of mathematics…*

I intended first to set up a formal system which comes ***as close as possible to actual reasoning****. The result was a* ***calculus of natural deduction****.*

Let’s now see the **DERIVATION RULES**

Reiteration



Rules for ∧

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\ \wedge\mathbf{I} \qquad\qquad \frac{\phi \wedge \psi}{\phi}\ \wedge\mathbf{E} \qquad \frac{\phi \wedge \psi}{\psi}\ \wedge\mathbf{E}$$

Derivation rules are introduced for the different connectives and there are **introduction rules** and **elimination rules** such as **∧I** and **∧E**

Rules for →

$$\frac{\phi \quad \phi\rightarrow\psi}{\psi}\ \rightarrow\mathbf{E}$$

$$\frac{\begin{matrix}[\phi]^{i} \\ \vdots \\ \psi\end{matrix}}{\phi\rightarrow\psi}\ \rightarrow\mathbf{I}^{i}$$

This is *modus ponens*

This rules says that if you **assume ϕ** and then **manage to derive** ψ, you can **derive** ϕ→ψ and **cancel assumption ϕ**

Some Mistakes

$$\dfrac{\begin{matrix}\phi \\ \vdots \quad \phi \to \psi\end{matrix}}{\psi}\ \to\mathbf{E}$$

WRONG use of →E!

$$\dfrac{\begin{matrix}\phi \\ \vdots \quad \psi\end{matrix}}{\phi \land \psi}\ \land\mathbf{I}$$

WRONG use of ∧I!

Conditional Proof and Rule →I

Claim: If if **n** is odd, $\mathbf{n^2}$ leaves a remainder of **1** when divided by **4**.

Suppose **n** is odd.

So, $\mathbf{n=2m+1}$ for some **m**. Then, by squaring **n**, we have:

$$\mathbf{n^2=(2m+1)^2=4m^2+4m+1=4(m^2+m)+1}$$

So, since $\mathbf{n^2=4(m^2+m)+1}$, when $\mathbf{n^2}$ is divided by **4**, it leaves a remainder of **1**.

Hence, if **n** is odd, $\mathbf{n^2}$ leaves a remainder of **1** when divided by **4**.

The Flexibility of Rule →I

The rule…

$$\dfrac{\begin{array}{c}[\phi]^{i}\\ \cdot\\ \cdot\\ \cdot\\ \psi\end{array}}{\phi\to\psi}\;\to\mathbf{I}^{i}$$

..and a sample derivation

$$\dfrac{\dfrac{\begin{array}{c}[p]^{1}\\ [q]^{2}\end{array}}{p\to q}\;\to\mathbf{I}^{1}}{q\to(p\to q)}\;\to\mathbf{I}^{2}$$

Two Somewhat Trivial Claims

Idempotency $\vdash \phi \rightarrow (\phi \wedge \phi)$ **and** $\vdash (\phi \wedge \phi) \rightarrow \phi$

Commutativity $\vdash (\phi \wedge \psi) \rightarrow (\psi \wedge \phi)$ **and** $\vdash (\psi \wedge \phi) \rightarrow (\phi \wedge \psi)$

Establishing $\vdash \phi \rightarrow (\phi \wedge \phi)$

$$\dfrac{\dfrac{[\phi]^1 \qquad [\phi]^1}{\phi \wedge \phi}\ \wedge\mathbf{I}}{\phi \rightarrow (\phi \wedge \phi)}\ \rightarrow\mathbf{I}^1$$

The same assumption $\boldsymbol{\phi}$ is used twice, although both instances are cancelled at once by one application of $\rightarrow$**I**

Establishing $\vdash (\phi \wedge \psi) \rightarrow (\psi \wedge \phi)$

$$\dfrac{\dfrac{\dfrac{[\phi \wedge \psi]^1}{\psi}\,\wedge\mathbf{E} \qquad \dfrac{[\phi \wedge \psi]^1}{\phi}\,\wedge\mathbf{E}}{\psi \wedge \phi}\,\wedge\mathbf{I}}{(\phi \wedge \psi) \rightarrow (\psi \wedge \phi)}\,\rightarrow\mathbf{I}^1$$

Summary of the Rules Thus Far

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\,\wedge\mathbf{I}$$

$$\frac{\phi \wedge \psi}{\phi}\,\wedge\mathbf{E}$$

$$\frac{\phi \wedge \psi}{\psi}\,\wedge\mathbf{E}$$

$$\frac{\phi}{\phi}\,\mathbf{R}$$

$$\begin{array}{c} [\phi]^{i} \\ \vdots \\ \psi \\ \hline \phi \to \psi \end{array}\;\to\mathbf{I}^{i}$$

$$\frac{\phi \qquad \phi \to \psi}{\psi}\,\to\mathbf{E}$$