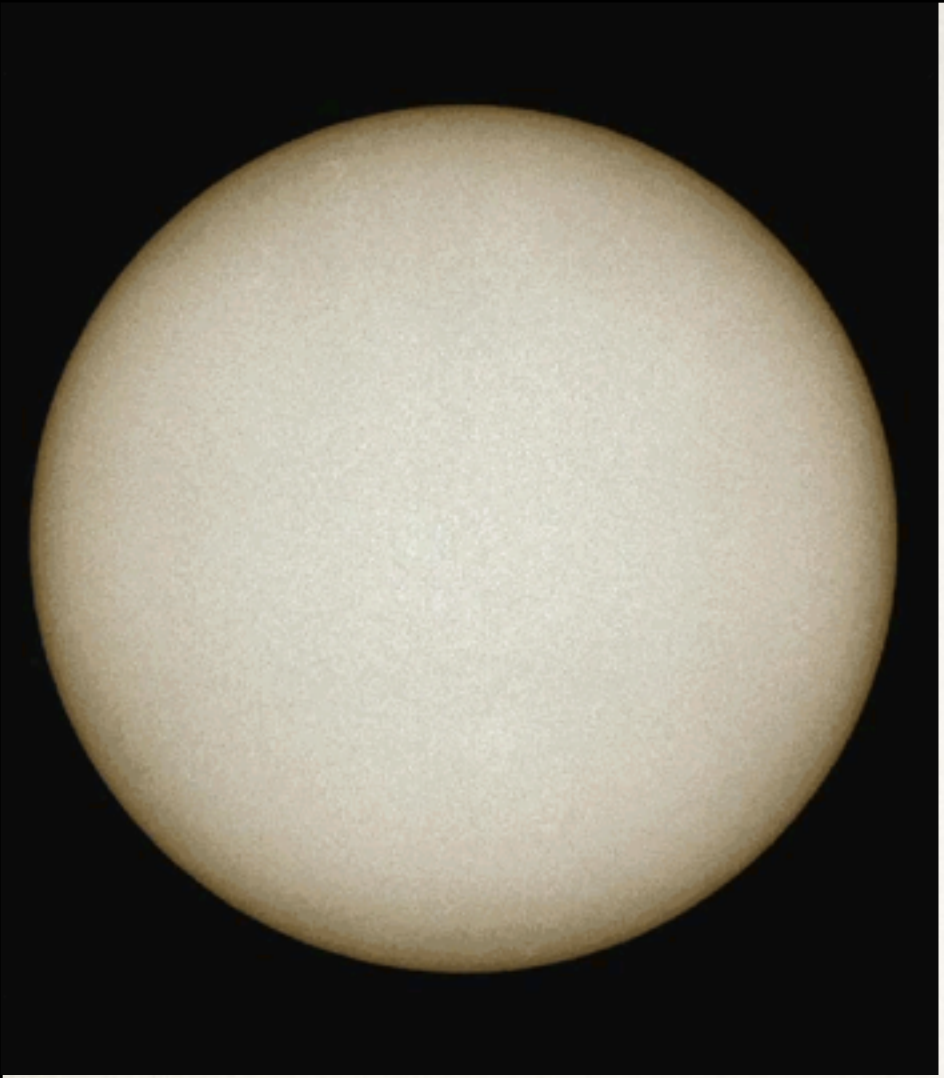
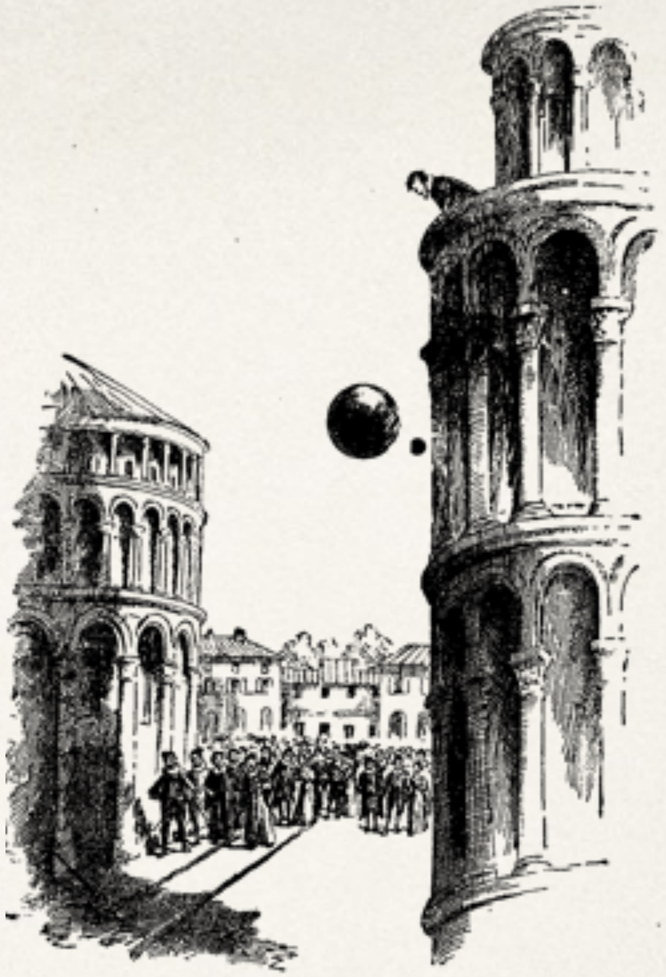

The One


Brouwer


B+S

PHIL 50 - Introduction to Logic

Marcello Di Bello, Stanford University, Spring 2014

Week 3 — Wednesday Class - Derivations in Propositional Logic (CONTINUED)

Summary of Monday's Rules

$$\frac{\phi}{\phi}\,\mathbf{R}$$

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\,\wedge\mathbf{I}$$

$$\frac{\phi \wedge \psi}{\phi}\,\wedge\mathbf{E}$$

$$\frac{\phi \wedge \psi}{\psi}\,\wedge\mathbf{E}$$

$$\frac{\begin{matrix}[\phi]^{i} \\ \vdots \\ \psi\end{matrix}}{\phi \rightarrow \psi}\,\rightarrow\mathbf{I}^{i}$$

$$\frac{\phi \rightarrow \psi \quad \phi}{\psi}\,\rightarrow\mathbf{E}$$

A Peculiarity of Our Propositional Language

Notational Convention:

We shall consider negated formulas of the form

$$\neg\phi$$

as abbreviations of

$$\phi\to\perp$$

We can convince ourselves that this notational convention is semantically plausible by looking at the truth tables for $\neg\phi$ and $\phi\to\perp$.

An Application of →E

$$\frac{\phi \quad \neg\phi}{\bot}\ {\to}\mathbf{E}$$

Given
our notational
convention, this is a
correct application of
rule →**E**

$$\frac{\phi \quad \phi\to\bot}{\bot}\ {\to}\mathbf{E}$$

An Application of →I

$$\dfrac{\begin{array}{c}[\phi]^i \\ \vdots \\ \bot\end{array}}{\neg\phi}\;\to\mathbf{I}^i$$

$$\dfrac{\begin{array}{c}[\phi]^i \\ \vdots \\ \bot\end{array}}{\phi\to\bot}\;\to\mathbf{I}^i$$

Given
our notational convention,
this is a correct application
of rule

Deriving the PNC

$$\cfrac{\cfrac{\cfrac{[\phi \wedge \neg \phi]^1}{\phi}\,\wedge\mathbf{E} \qquad \cfrac{[\phi \wedge \neg \phi]^1}{\neg \phi}\,\wedge\mathbf{E}}{\bot}\,\to\mathbf{E}}{\neg(\phi \wedge \neg \phi)}\,\to\mathbf{I}^1$$

Note the use of our notational convention in the application of rules **→E** and **→I**

Now Let’s See Some New Rules

Rules for ⊥

$$\frac{\perp}{\psi}\ \perp$$

$$\frac{\begin{array}{c}[\neg\phi]^i \\ \vdots \\ \perp\end{array}}{\phi}\ \mathbf{RAA}^i$$

This rule formalizes the thesis that **from the contradiction anything follows**.

This rules formalizes **proof by contradiction**. **RAA** is an abbreviation of the Latin expression ***reductio ad absurdum***.

From the Contradiction Anything Follows (*ex contradictione quodlibet*)

Does the rule make sense?

Yes!
Semantically, the rule makes sense

$$\frac{\perp}{\psi}\,\perp$$

(ϕ	$\wedge$	$\neg$	ϕ)		ψ
1	0	0	1		…
0	0	0	o		…

We can always write $\perp \vDash \psi$ no matter what the truth value of ψ turns out to be because holds $\perp \vDash \psi$ vacuously.

Proof by Contradiction (*reductio ad absurdum*)

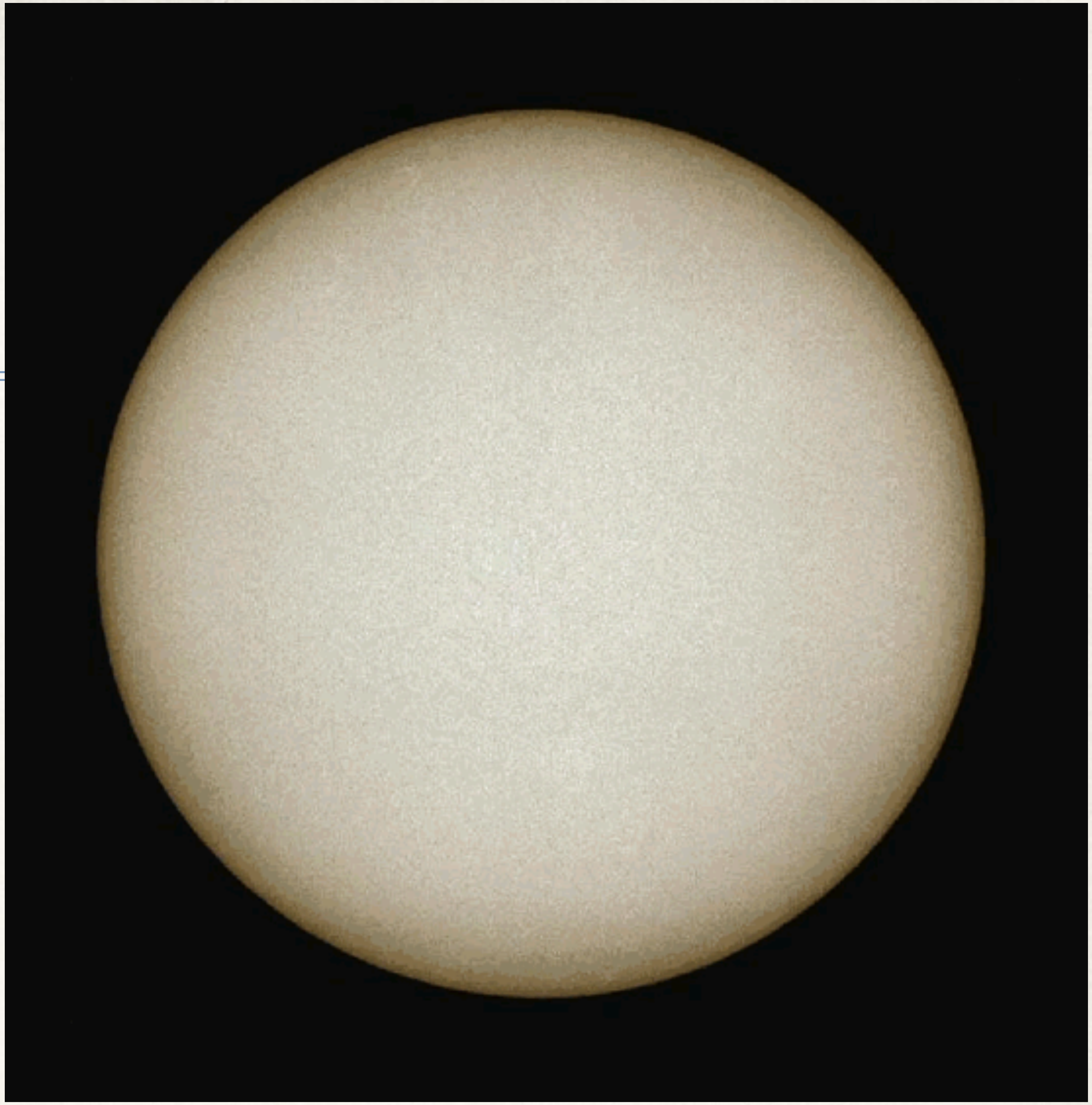
$$\dfrac{\begin{matrix}[\neg\phi]^i \\ \vdots \\ \bot\end{matrix}}{\phi}\ \textbf{RAA}^i$$

The idea behind this form of reasoning is that you can **establish a positive claim ϕ by showing that the negation of ϕ leads to a contradiction.**

This is a form of **indirect proof** because you do not establish **ϕ** directly but by showing that its negation implies a contradiction.

Proof by Contradiction:
Zeno of Elea

- *Suppose MANY THINGS EXIST.*
- *If they are many, they will be as many as they are, no more and no fewer. Thus, they will be finite.*
- *If there is a finite number of things, there will be an infinite number of things, because something will exist between two things, and so on.*



- *So, there will be a finite and an infinite number of things. Contradiction.*
- *Hence, ALL IS ONE.*

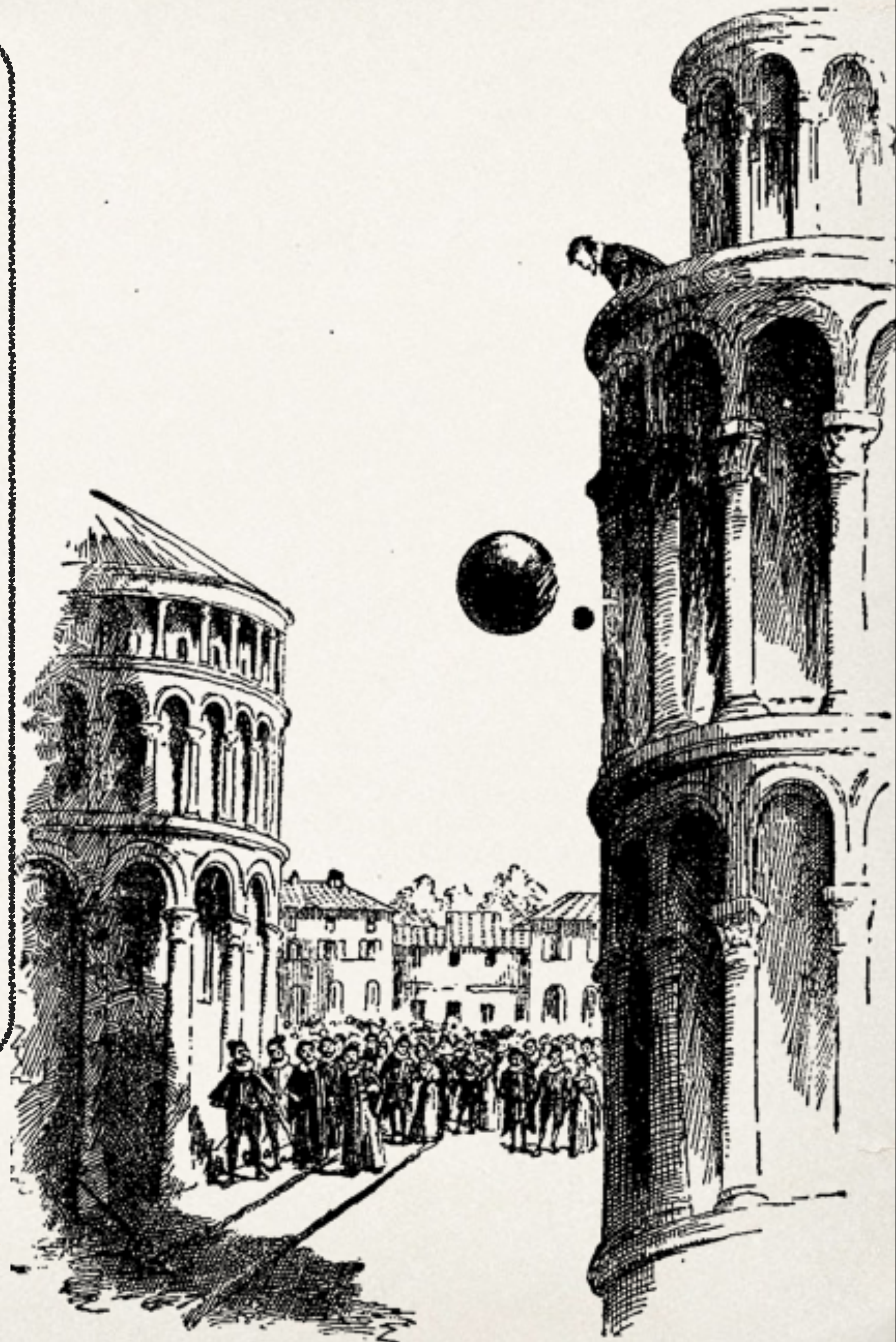
Galileo's Critique of Aristotle

Aristotle thinks that heavier bodies fall faster than lighter ones, i.e. speed is proportional to weight (other things being equal).

Take a small and a bigger body, **S** and **B**. If they are combined, **S** will slow down **B**, so **S+B** will fall slower than **B** alone. But **S+B** is heavier than **B**, so **S+B** must fall faster.

S+B must fall slower **and** faster than **B** alone. Contradiction!

What argument is this? RAA?
That depends on the conclusion we draw from the contradiction.



Do Not Confuse →I with RAA

$$\dfrac{\begin{matrix}[\phi]^i \\ \vdots \\ \bot\end{matrix}}{\neg\phi}\ \to\mathbf{I}^i$$

$$\dfrac{\begin{matrix}[\neg\phi]^i \\ \vdots \\ \bot\end{matrix}}{\neg\neg\phi}\ \to\mathbf{I}^i$$

$$\dfrac{\begin{matrix}[\neg\phi]^i \\ \vdots \\ \bot\end{matrix}}{\phi}\ \mathbf{RAA}^i$$

Going from $\neg\neg\phi$ to ϕ is not obvious!

Establishing $\vdash (\neg\neg\phi \rightarrow \phi)$

$$\dfrac{\dfrac{\dfrac{[\neg\neg\phi]^1 \quad [\neg\phi]^2}{\bot}\ \rightarrow\mathbf{E}}{\phi}\ \mathbf{RAA}^2}{(\neg\neg\phi\rightarrow\phi)}\ \rightarrow\mathbf{I}^1$$

The formula $\neg\neg\phi\rightarrow\phi$ says that **two negations make an affirmation.**

The derivation of $\neg\neg\phi\rightarrow\phi$ crucial rests upon **RAA**

Intuitionistic logic

Those who deny **RAA** or principles like $\neg\neg\phi \rightarrow \phi$ are called **intuitionistic logicians**.

They believe that in mathematics there should be no indirect proofs, but only direct ("constructive") proofs.

Brouwer