$\vdash \Rightarrow \models$

Soundness

$\models \Rightarrow \vdash$

Completeness

PHIL 50 - Introduction to Logic

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Derivability: $\vdash$

$\vdash \psi$ *iff*

there is a derivation of ψ in which all assumptions are canceled.

$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$ *iff*

there is a derivation of ψ from assumptions $\phi_1, \phi_2, \ldots, \phi_k$

A derivation is a tree-like arrangement of formulas which obeys the derivation rules we studied during Week 3 of the course.

Logical Consequence: $\models$

$\models \psi$ *iff*

all valuation V's make ψ true

$\phi_1, \phi_2, \ldots, \phi_k \models \psi$ *iff*

all valuations V's that make $\phi_1, \phi_2, \ldots, \phi_k$ true make also ψ true

We studied the notion of logical consequence during Week 2 of the course

The Equivalence of $\vdash$ and $\vDash$ in Propositional Logic

$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$

SOUNDNESS $\Downarrow$ $\Uparrow$ COMPLETENESS

$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$

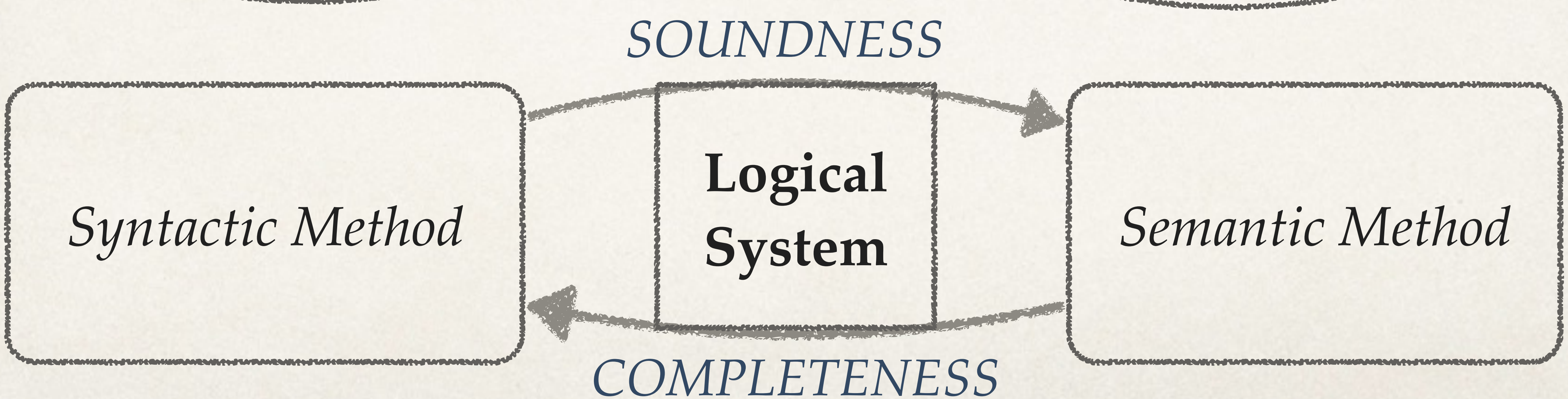
Two (Logical) Ways to Identify Good Arguments

Check whether the proposed argument **conforms to a derivation**

$\vdash$

Check whether **all the valuations that make true all the premises make true the conclusion as well**

$\vDash$



Why Does the **SOUNDNESS** of Propositional Logic Matter?

$$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$$

SOUNDNESS $\Downarrow$

$$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$$

Soundness as “Semantic Check”

How do we know that the
derivation rules we have chosen are **good rules**?

Soundness *guarantees that the* ***derivation rules*** *we have chosen are* ***truth preserving*** *(i.e. they always bring us from true premises to true conclusions)*

Derivations rules

$$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$$
$$\Downarrow$$
$$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$$

Truth preservation

The Example of Rule RAA

Is RAA a good derivation rule to have?

$$\dfrac{\begin{array}{c}[\neg\phi]^i \\ \vdots \\ \bot\end{array}}{\phi}\ \mathbf{RAA}^i$$

IF $\neg\phi \vDash \bot$ THEN $\vDash \phi$

$\neg\phi$	$\bot$
0	0
~~1~~	~~0~~

ϕ
1
~~0~~

IF $\neg\phi \vdash \bot$ THEN $\vdash \phi$

*This provides a **"semantic check"** on rule **RAA***

What Happens if We Allow for **Three Truth Values?**

An Example: Is **RAA** Still Good?

Is RAA a good derivation rule to have?

$$\dfrac{\begin{array}{c}[\neg\phi]^i \\ \vdots \\ \bot\end{array}}{\phi}\ \mathbf{RAA}^i$$

$\neg\phi \vDash \bot$ BUT $\nvDash \phi$

$\neg\phi$		$\bot$
0		0
~~1~~		~~0~~
0.5		0

ϕ
1
~~0~~
0.5

$\neg\phi \vdash \bot$ BUT $\nvdash \phi$

In a three valued semantics ***RAA*** *is no longer good*

<u>Upshot</u>: Soundness is not Absolute; it is Relative to a Given Semantics

Why Does the **COMPLETENESS** of Propositional Logic Matter?

$$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$$

$$\Uparrow \quad \textbf{COMPLETENESS}$$

$$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$$

The Clever Is Reduced to the Automatic

DIFFICULT TASK

Syntactic method to identify good arguments:

Construct **a derivation**

COMPLETENESS

Semantic method to identify good arguments:

Fill out **a truth table** appropriately

AUTOMATIC TASK

Completeness *of propositional logic allows ut to reduce a task that requires some* ***cleverness*** *(i.e. constructing derivations) with a task that is* ***completely automatic*** *(i.e. constructing truth-tables)*

Is This Formula Derivable?

$((\phi \to \psi) \to \phi) \to \phi$

$((\phi$	$\to$	$\psi)$	$\to$	$\phi)$	$\to$	ϕ
1	1	1	1	1	1	1
1	0	0	1	1	1	1
0	1	1	0	0	1	0
0	1	0	0	0	1	0

By the truth table method we know that

$$\models ((\phi \to \psi) \to \phi) \to \phi$$

and by completeness we know that

$$\vdash ((\phi \to \psi) \to \phi) \to \phi$$

Easy!

Can You Find the Derivation?

$$
\dfrac{\dfrac{\dfrac{\dfrac{[((\phi\to\psi)\to\phi)]^3 \qquad \dfrac{\dfrac{\dfrac{[\neg\phi]^1 \qquad [\phi]^2}{\bot}\,{\to}\mathbf{E}}{\psi}\,\bot}{\phi\to\psi}\,{\to}\mathbf{I}^2}{\phi}\,{\to}\mathbf{E} \qquad [\neg\phi]^1}{\bot}\,{\to}\mathbf{E}}{\phi}\,\mathbf{RAA}^1}{((\phi\to\psi)\to\phi)\to\phi}\,{\to}\mathbf{I}^3
$$

Not So Easy!

The Power of Completeness — if $\vDash$ then $\vdash$

How do we know that the derivation rules we have chosen are **ALL the derivations rules we need**? Maybe we need more?

Completeness** guarantees that the derivation rules we have chosen are **ALL the derivation rules we need

Finite number of derivations rules

$$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$$

$$\Uparrow$$

$$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$$

All possible logical consequences

Summary of Soundness and Completeness

Soundness *guarantees that the* ***derivation rules*** *we have chosen are good rules insofar as they are* ***truth-preserving*** *(i.e. they always bring us from true premises to true conclusions)*

Completeness *guarantees that the* ***derivation rules*** *we have chosen are* ***ALL the derivation rules we need****. No extra derivation rules are needed.*

Two Equivalent Formulations

$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$

SOUNDNESS $\Downarrow$ $\Uparrow$ COMPLETENESS

$\phi_1, \phi_2, \ldots, \phi_k \models \psi$

$\phi_1, \phi_2, \ldots, \phi_k \not\models \psi$

SOUNDNESS $\Downarrow$ $\Uparrow$ COMPLETENESS

$\phi_1, \phi_2, \ldots, \phi_k \not\vdash \psi$

How Do We Prove Soundness and Completeness?

You will have to take a more advanced logic course (e.g. **PHIL 150** *or* **PHIL 151***) to see how the proof goes.*

Consistency

A set $\mathbf{\Gamma}$=$\{\phi_1, \phi_2, \ldots, \phi_k\}$ is consistent

iff

$\phi_1, \phi_2, \ldots, \phi_k \nvdash \perp$

Completeness, Soundness and Consistency

$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$

SOUNDNESS $\Downarrow$ $\Uparrow$ COMPLETENESS

$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$

If COMPLETENESS and SOUNDNESS hold, then

the set $\Gamma = \{\phi_1, \phi_2, \ldots, \phi_k\}$ is consistent

iff

there is a valuation **V** that makes all formulas in Γ true

To Establish an IFF-Claim We Should Prove Both Directions

If *COMPLETENESS* and SOUNDNESS hold, then the set $\mathbf{\Gamma=\{\phi_1, \phi_2, \ldots, \phi_k\}}$ is consistent

$$\Rightarrow$$

there is a valuation **V** that makes all formulas in **Γ** true

PROOF:

Suppose $\mathbf{\Gamma=\{\phi_1, \phi_2, \ldots, \phi_k\}}$ is consistent, which means that $\boldsymbol{\phi_1, \phi_2, \ldots, \phi_k} \nvdash \bot$.

By completeness, it follows that $\boldsymbol{\phi_1, \phi_2, \ldots, \phi_k} \nvDash \bot$.

Now, $\boldsymbol{\phi_1, \phi_2, \ldots, \phi_k} \nvDash \bot$ means that there is a valuation **V** such that **V** makes true $\boldsymbol{\phi_1, \phi_2, \ldots, \phi_k}$ and **V** does not make true $\bot$.

So, there is a **V** that makes true all formulas in **Γ**.

If *COMPLETENESS and SOUNDNESS* hold, then the set $\mathbf{\Gamma=\{\phi_1, \phi_2, \ldots, \phi_k\}}$ is consistent

$\Leftarrow$

there is a valuation **V** that makes all formulas in $\mathbf{\Gamma}$ true

PROOF:

Suppose there is a **V** that makes true all formulas in $\mathbf{\Gamma}$.

By definition **V** does not make $\perp$ true.

So, there is a **V** that makes true all formulas in $\mathbf{\Gamma}$ and does not make $\perp$ true. In other words, $\mathbf{\phi_1, \phi_2, \ldots, \phi_k} \not\models \perp$.

By soundness, $\mathbf{\phi_1, \phi_2, \ldots, \phi_k} \nvdash \perp$, so $\mathbf{\Gamma}$ is consistent.