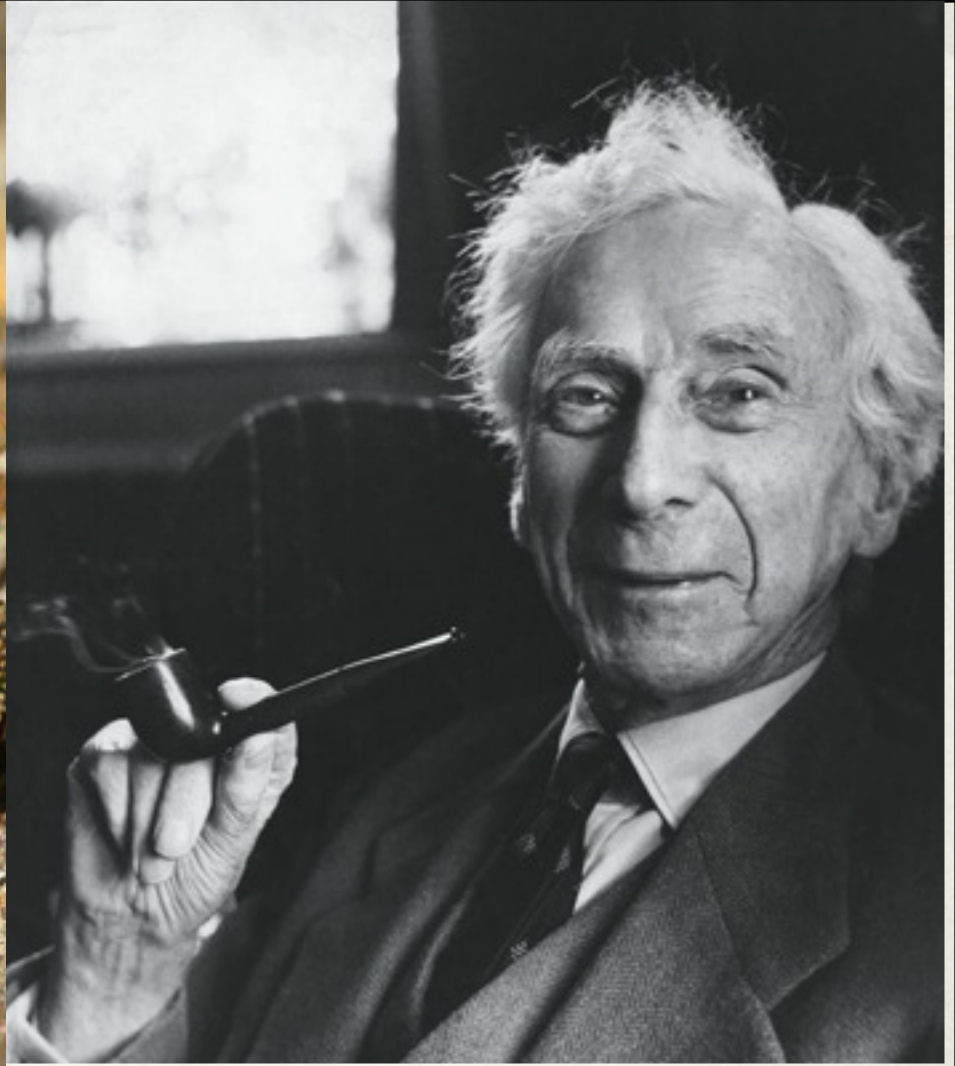
Aristotle

A Lizard



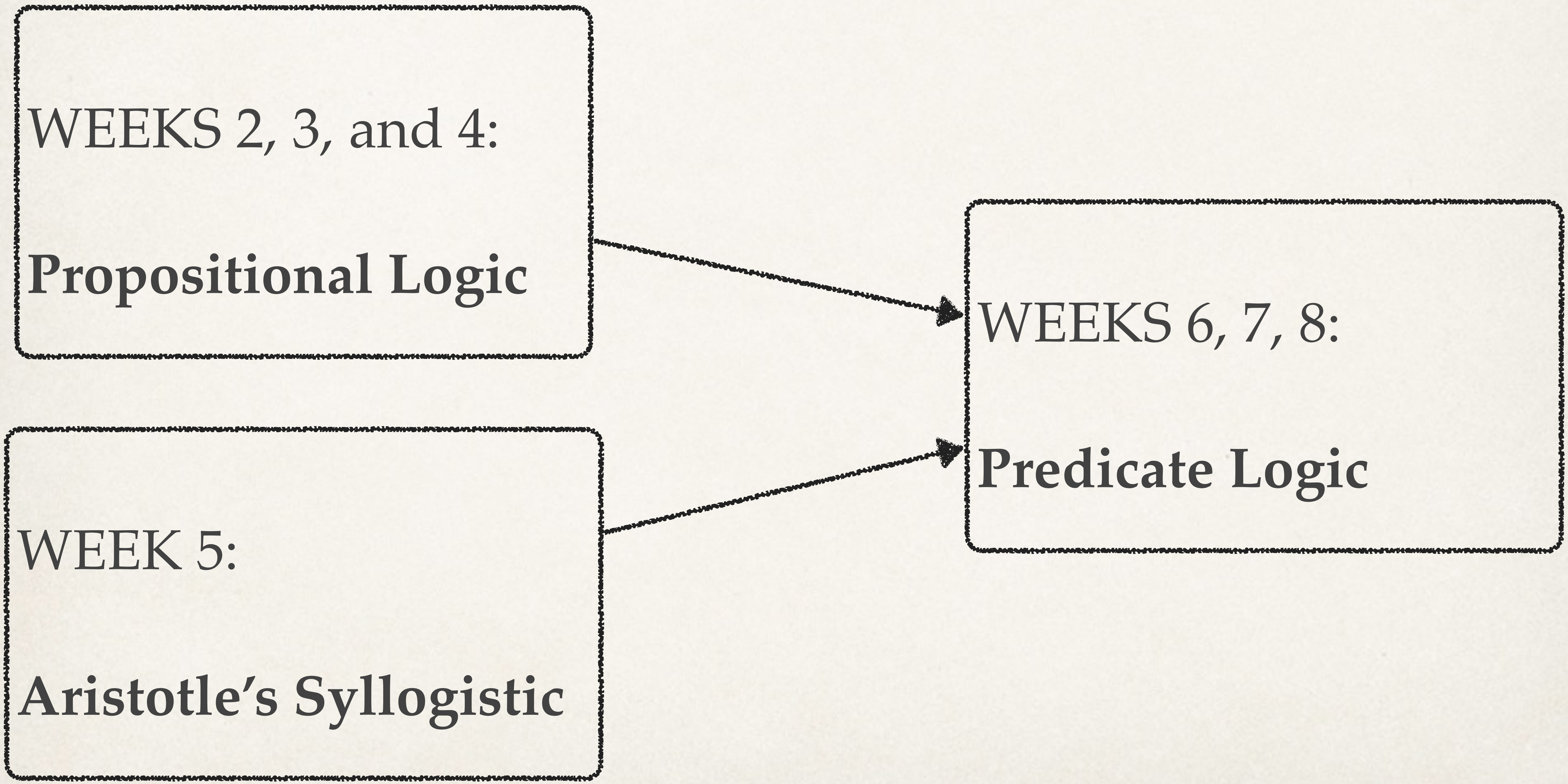
Bertrand Russell

PHIL 50 - Introduction to Logic

Marcello Di Bello, Stanford University, Spring 2014

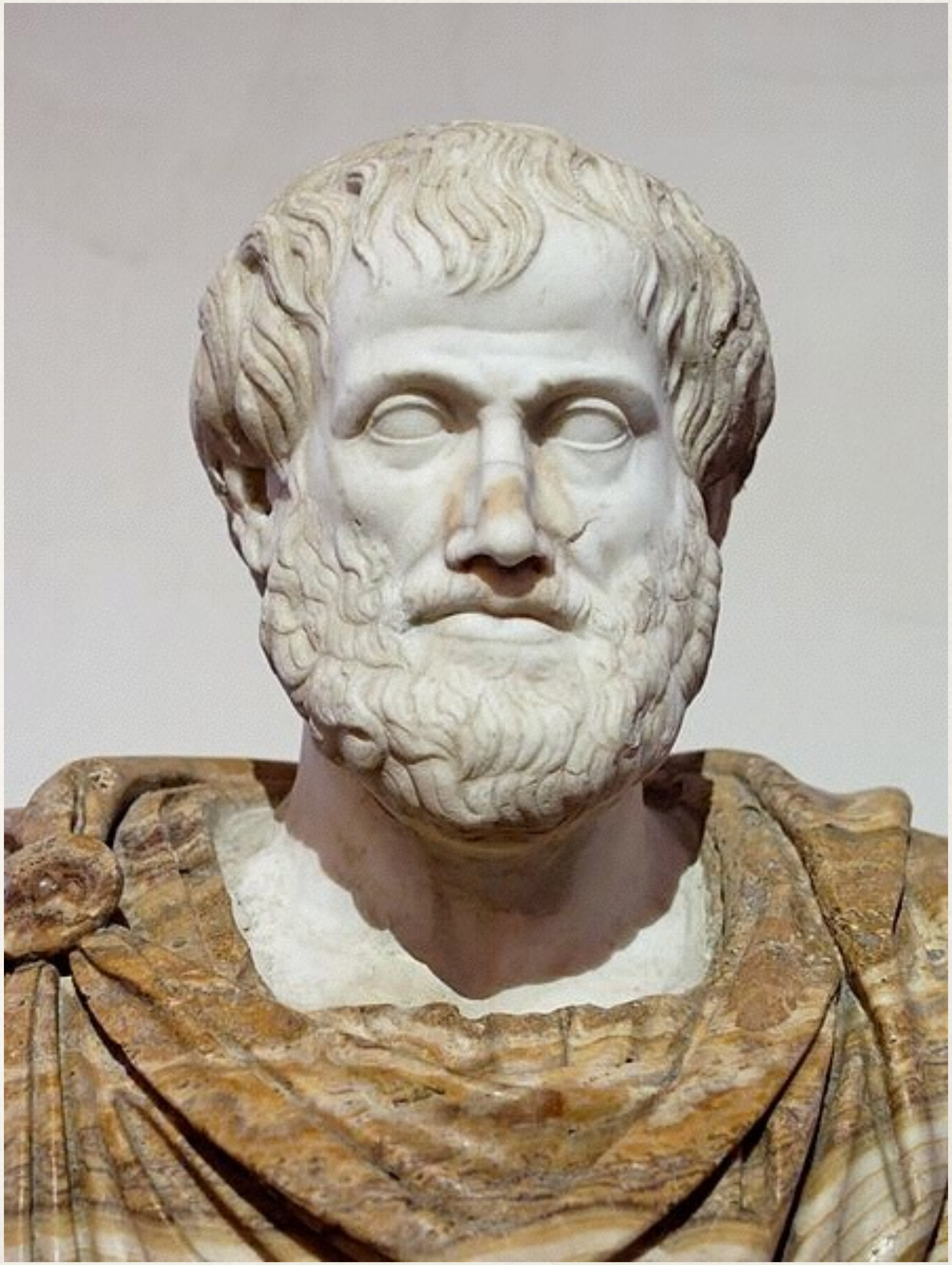
Week 5 — Monday Class - Syllogistic Logic and Sets

Where We Are in the Course



Syllogistic Logic

Aristotle expounds his theory of the syllogism in the treatise *Prior Analytics*.



Syllogism (1)

All animals are mortal
All humans are animals

All humans are mortal

All *B* are *C*
All *A* are *B*

All *A* are *C*

Syllogism (2)

(Suggested by Lewis Carroll)

No bald creature is in need of an airbrush
All lizards are bald creatures

No lizard is in need of an airbrush

No A is B
All C are A

No C is B

Syllogism (3)

All tomatoes are rotten
Some chickpeas are **not** rotten

No chickpeas are tomatoes

All A is B
Some C are **not** B

No C is A

NB: Not all syllogism have to be valid. Some will be valid and some will be invalid. This one looks like an invalid one, right? ***More on this later.***

What is a Syllogism?

An argument with two premises and one conclusion

Premises and conclusion can take the following forms:

All A are B

Some A are B

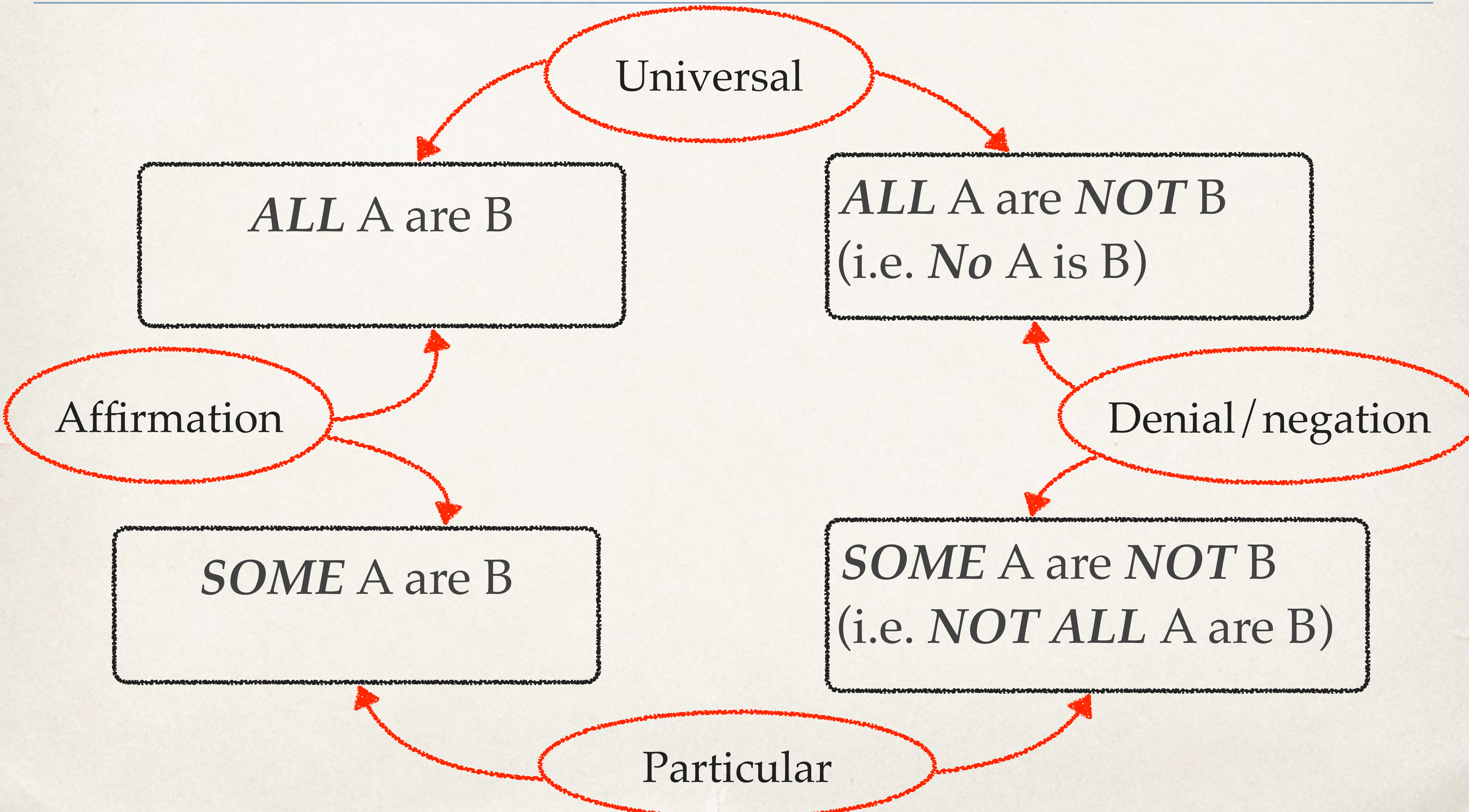
All A are **not** B (i.e. **No** A is B)

Some A are **not** B (i.e. **Not all** A are B)

A and **B** are **predicates** and denote **sets** (*more later*).

A syllogism contains at most 3 different predicates

Aristotle's Classification of Statements



Four Types of Statements

Universal affirmative	**All** lizards are bald creatures
Universal negative	**All** lizards are **not** bald creatures (i.e. **No** lizard is a bald creature)
Particular affirmative	**Some** lizards are bald creatures
Particular negative	**Some** lizards are **not** bald creatures (i.e. **Not all** lizards are bald creatures)

How Many Syllogistic Patterns Are There ?

Since a syllogism contains **three statements**, and of each statement there are **four types**, in total there are

4x4x4=64

syllogistic patterns.

But which of those 64 syllogistic patterns are valid?

In other words, which of those 64 syllogistic patterns always bring us from true premises to a true conclusion?

Checking Validity

How Do We Know These Syllogistic Patterns Are Valid or Invalid?

All *A* are *B*
All *B* are *C*

All *A* are *C*

No *A* is *B*
All *C* are *A*

No *C* is *B*

All *A* are *B*
Some *C* are **not** *B*

No *C* is *A*

Since **A, B** and **C** are **predicates** and they denote **sets**, an excursus into the **nature of sets** and **their operations** is needed at this point.

Sets

The way we will check the validity of syllogistic patterns is not due to Aristotle. Set theory was not yet developed at the time of Aristotle.

Predicates and Sets

Predicates are linguistic expressions. We will use upper case letters, such as **A, B, C**, to represent generic predicate.

Sets of objects correspond to the meaning of the predicates, or in other words, sets of objects are what predicates refer to.

Language

Meaning

Predicates and Sets: Bananas

Predicate/linguistic expression

Set of bananas/meaning

“is a banana”

Predicates and Sets: Coins

Predicate/linguistic expression

Set of coins/meaning

"is a coin"

What is a Set?

A set is any collection of objects/entities

Examples:

the set of bananas
the set of coins
the set of people who get up at 12 noon
the set of all natural numbers

Elements of a Set

If an element **a** belongs to a set **A**, we will write **a** ∈ **A**

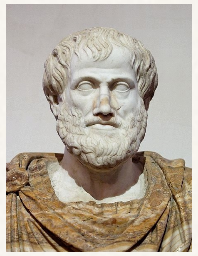
Observation 1: Elements of sets are denoted by lower case letters **a**, **b**, **c**, …

Observation 2: Sets are denoted by upper case letters **A**, **B**, **C**, …

Observation 3: Predicates are also denoted by upper case letters **A**, **B**, **C**, … , but keep in mind that *predicates and sets are different*. Predicates are linguist expressions referring to sets.

Defining Sets (1)

You can define a set, call it **A**, by listing all its elements between curly brackets, as follows:

A={  ,  ,  }

But what to do if the set is **very big** or **even infinite**?

Defining Sets (2)

You can define the set **B** of, say, bananas as follows:

$$B=\{x \mid x \text{ is a banana}\}$$

The definition says that **B** is the set containing all elements **x** such that **x** is a banana. In other words,

$$x \in B \;\; \textit{iff} \;\; x \text{ is a banana}$$

We can say that  $\in$ **B**

Defining Sets (3)

More generally, you can define a set, call it **A**, as follows:

$$\mathbf{A=\{x \mid x \text{ is } P\}}$$

The definition says that **A** is the set containing all elements **x** such that **x** is **P**, where **P** is whatever predicate (e.g. "is a banana", "is a morning person", etc.). In other words,

$$\mathbf{x \in A} \textit{ iff } \mathbf{x \text{ is } P}$$

You can also write, more succinctly, as follows:

$$\mathbf{A=\{x \mid P(x)\}}$$

Notation

Both sets and predicates are denoted by upper case letters such as **A**, **B**, **C**, …

Do not confuse sets of objects with the predicates referring to such sets.

The context should make it clear when **A**, **B**, **C**, … stand for sets and when **A**, **B**, **C**, … stand for predicates.

Sets and subsets

Subsets: $\subseteq$ and $\subset$

Whenever all the elements of one set **A** are also contained in a set **B**, then **A** is a subset of **B**. *In symbols*: $\mathbf{A} \subseteq \mathbf{B}$

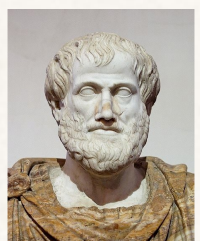
If all the elements of **A** are contained in **B**, and all the elements of **B** are contained in **A**, then **A** and **B** are the same set. *In symbols*: $\mathbf{A}=\mathbf{B}$

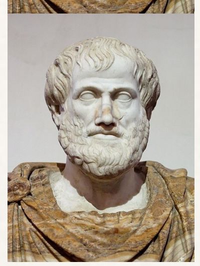
Note that *if* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \subseteq \mathbf{A}$, *then* $\mathbf{A}=\mathbf{B}$.

Whenever all the elements of one set **A** are also contained in a set **B**, and **B** contains at least one element which **A** does not contain, **A** is a proper subset of **B**. *In symbols*: $\mathbf{A} \subset \mathbf{B}$

Examples (1)

Let

A={  ,  ,  **}**

B={  ,  **}**

C= { }

B $\subset$ A and B $\subseteq$ A

C $\subset$ B and C $\subseteq$ B

C $\subset$ A and C $\subseteq$ A

C= { } is the empty set, also denoted by $\varnothing$

Examples (2)

Let

O={ x | X is an odd number }

E={ x | x is an even number }

N={ x | x is a natural number

O $\not\subset$ E and O $\nsubseteq$ E

E $\not\subset$ O and E $\nsubseteq$ O

O $\subset$ N and O $\subseteq$ N

E $\subset$ N and E $\subseteq$ N

The symbols $\not\subset$ and $\nsubseteq$ stand for **not-**$\subset$ and **not-**$\subseteq$

Returning to Syllogisms and Checking their Validity or Invalidity

From Statements to Sets (1)

Syllogism

All A are B
All B are C

All A are C

Set-theoretic translation

$\boldsymbol{A \subseteq B}$
$\boldsymbol{B \subseteq C}$

$\boldsymbol{A \subseteq C}$

Is the syllogism valid?

We need to check **whether the subset relation $\subseteq$ is transitive**. If it is transitive, we can conclude that the syllogism in question is valid.

The Transitivity of $\subseteq$

Suppose $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{C}$. We should prove that $\mathbf{A} \subseteq \mathbf{C}$.

Suppose (*) $\mathbf{A} \subseteq \mathbf{B}$ and (**) $\mathbf{B} \subseteq \mathbf{C}$.

In order to establish that $\mathbf{A} \subseteq \mathbf{C}$, we need to show that if $\mathbf{a} \in \mathbf{A}$, then $\mathbf{a} \in \mathbf{B}$, for <u>*an arbitrary element*</u> $\mathbf{a}$.

So, suppose $\mathbf{a} \in \mathbf{A}$. Since by assumption (*) $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{a} \in \mathbf{B}$. Further, since by assumption (**) $\mathbf{B} \subseteq C$, then $\mathbf{a} \in \mathbf{C}$.

So, if $\mathbf{a} \in \mathbf{A}$, then $\mathbf{a} \in \mathbf{C}$, and since $\mathbf{a}$ was arbitrary, $\mathbf{A} \subseteq \mathbf{C}$.

What Have we Really Accomplished?

Syllogism

Set-theoretic translation

All A are B
All B are C

All A are C

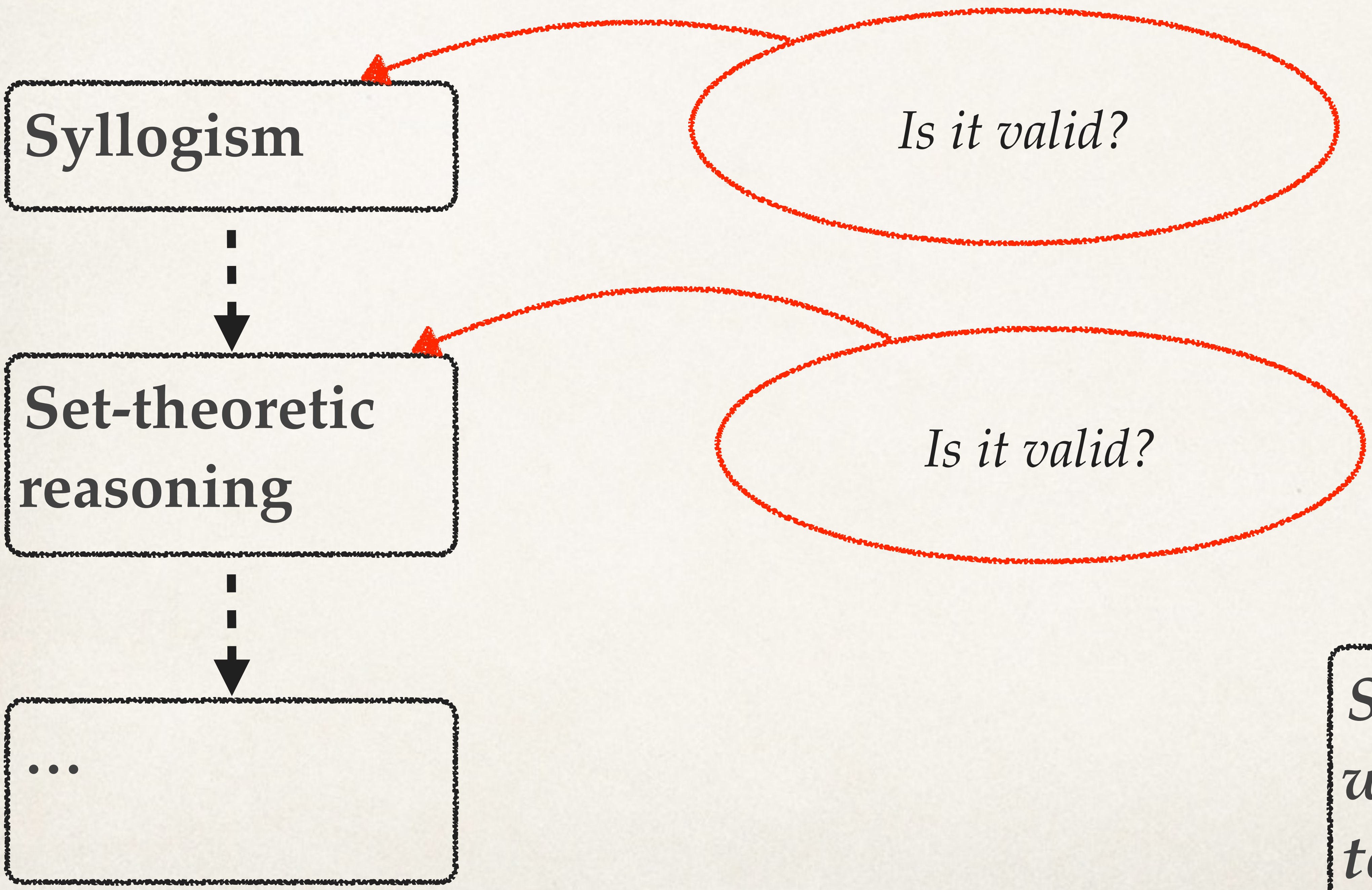
$\boldsymbol{A \subseteq B}$
$\boldsymbol{B \subseteq C}$

$\boldsymbol{A \subseteq C}$

We showed the validity of the syllogism by **relying on reasoning based on** $\subseteq$

But how do we know that the reasoning based on the subset relation $\subseteq$ *is itself valid?*

An Infinite Regress



Somewhere we will have to stop!

Is It Safe to Say That We Trust Reasoning Based On Sets?

Sets Can Get Wild!

Two *prima facie* plausible assumptions:

1. **A set can contain a set as one of its elements.** For example consider the set of bananas and the set of strawberries. Now, there can exist a set that contains both these sets. So, a set can be an element of a set.

2. **A set con contain itself as one of its elements.** For instance, consider the set of all sets. Since it is itself a set, it must be an element of the set of all sets, and thus it must be an element of itself.

The Set of All Sets That Are Not Elements of Themselves

Consider the set **R** such that $\mathbf{R=\{x \mid x \notin x\}}$.

In other words, **R** is the set of all sets that are not elements of themselves. There seem to be many sets that are not elements of themselves, e.g. the set of all elephants, the set of all ideograms, etc.

It is plausible to say that there is a set that collects all the sets that are not elements of themselves. Let's call such a set **R**.

Russell's Paradox

Now if $\mathbf{R}=\{\mathbf{x} \mid \mathbf{x} \notin \mathbf{x}\}$, then

$$(*) \qquad \mathbf{x} \in \mathbf{R} \text{ iff } \mathbf{x} \notin \mathbf{x}.$$

Let's now see whether $\mathbf{R} \in \mathbf{R}$ or $\mathbf{R} \notin \mathbf{R}$.

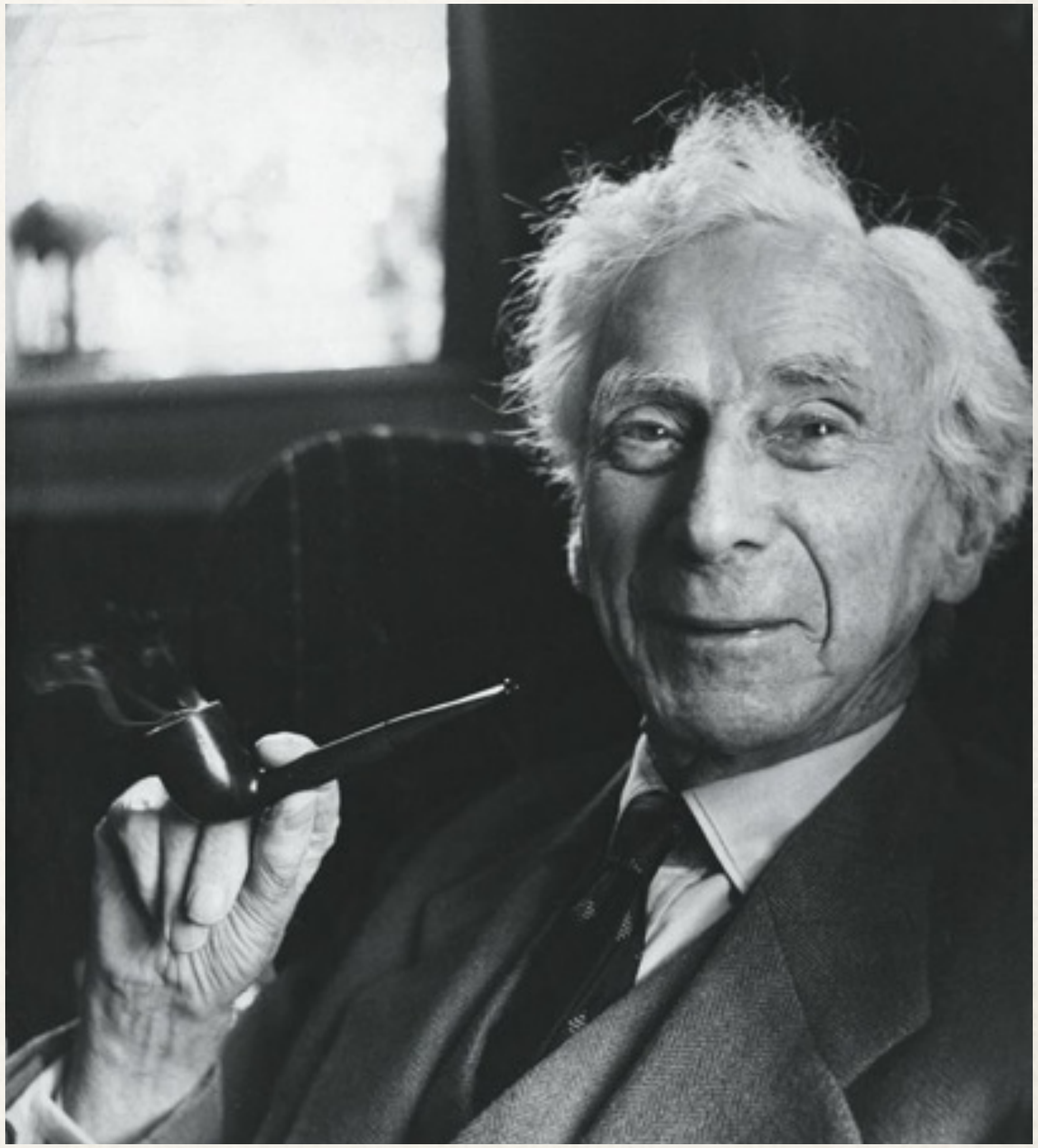
If $\mathbf{R} \in \mathbf{R}$ then $\mathbf{R} \notin \mathbf{R}$ by replacing $\mathbf{x}$ with $\mathbf{R}$ in **(*)**.

If $\mathbf{R} \notin \mathbf{R}$ then $\mathbf{R} \in \mathbf{R}$ by replacing $\mathbf{x}$ with $\mathbf{R}$ in **(*)**.

Either way, we get a contradiction.

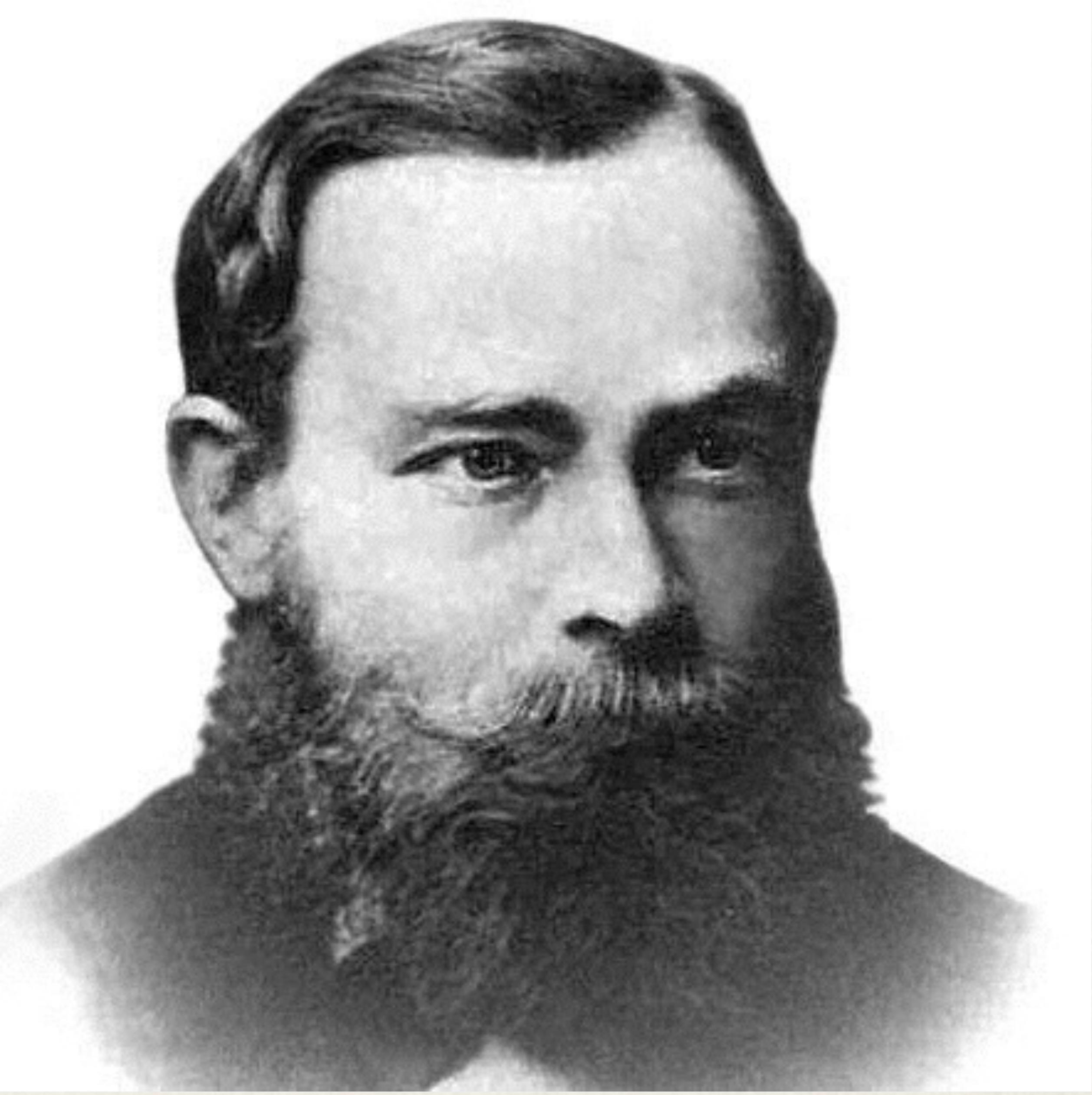
Russell's Letter to Frege

Bertrand Russell formulated his paradox in a letter in 1902 to Frege. Frege believed that logic and sets alone could constitute the foundation of mathematics. (*Frege's position is known as logicism*.)



Frege's Response to Russell

"Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetics…It is all the more serious because … not only the foundations of my arithmetics, but also the sole possible foundations of arithmetics, seem to vanish."



What to Do, Then?

Not everything which is an imaginable collection of objects can be a set. *Take Philosophy of Math if want to know more.*

<u>For our purposes</u>, it suffices to say that sets exist within a universe **U** of discourse which is itself a set and from which the elements of the set we are defining are taken. Instead of simply writing

B={x | x is a banana}

we should—strictly speaking—write:

B={x ∈ U | x is a banana}