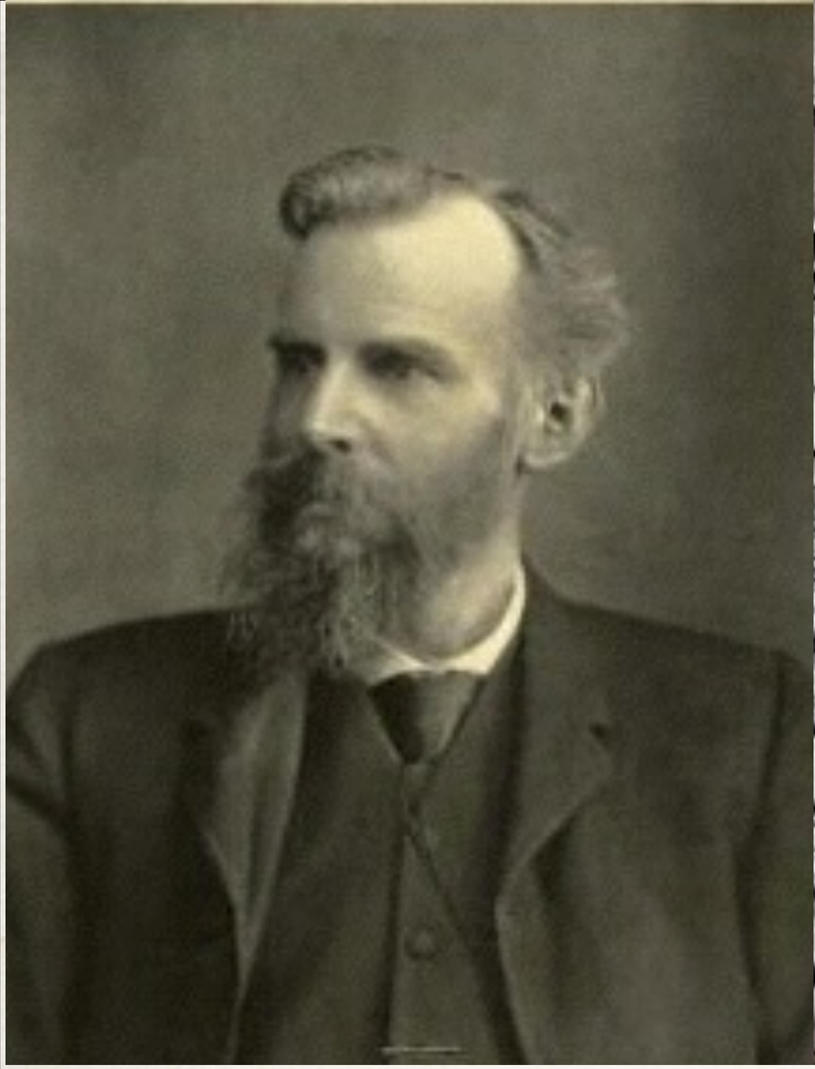
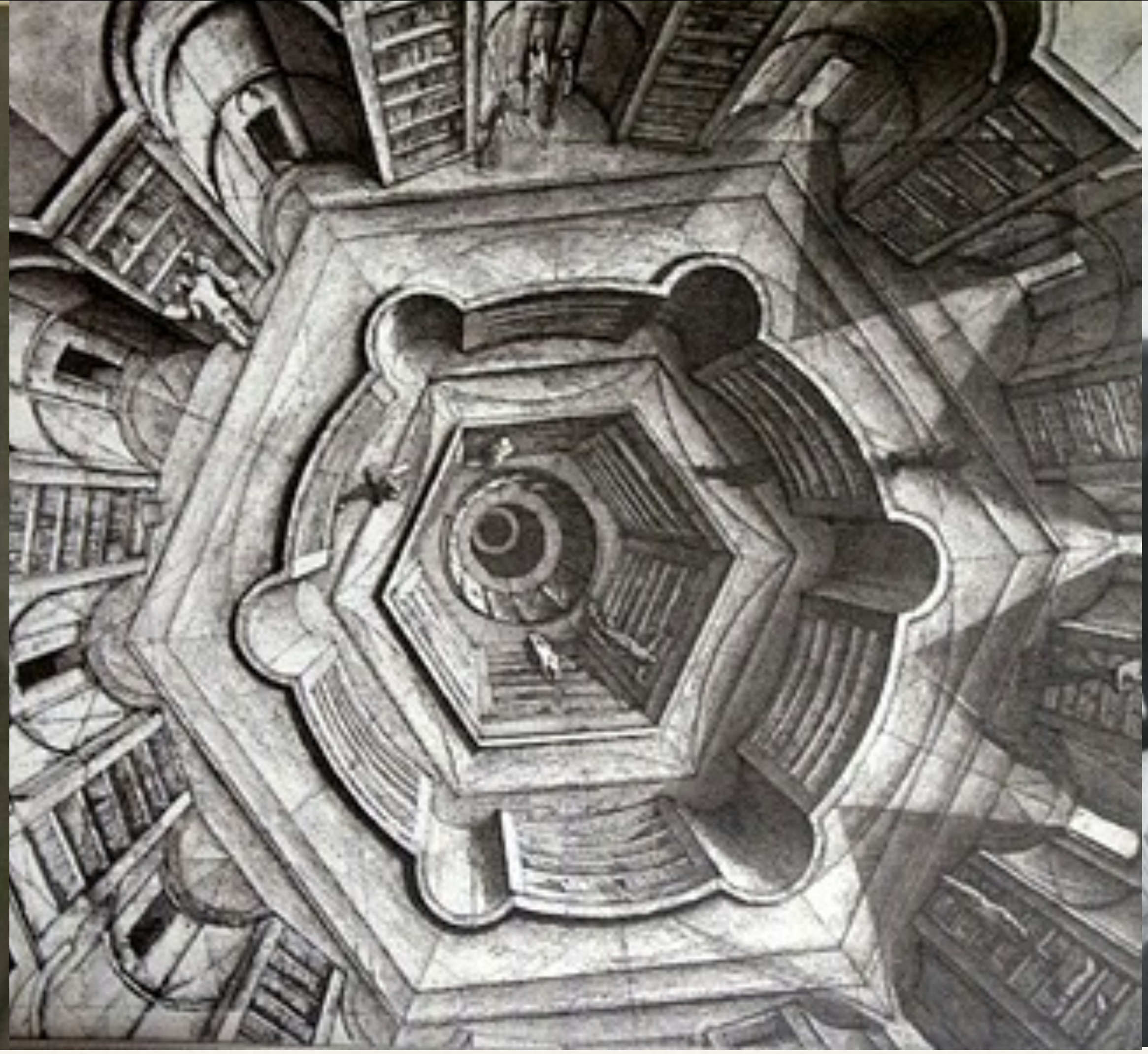
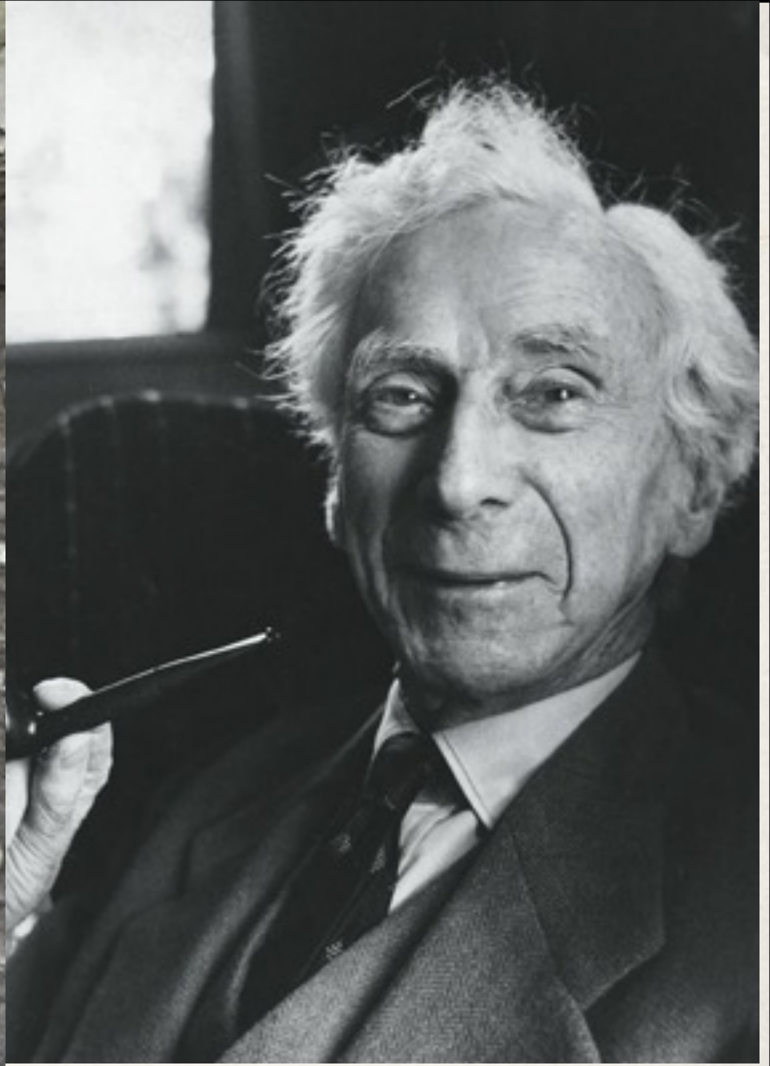


John Venn



The Library of Babylon



Bertrand Russell

PHIL 50 - Introduction to Logic

Marcello Di Bello, Stanford University, Spring 2014

Week 5 — Wednesday Class - Syllogistic Logic and Sets (II)

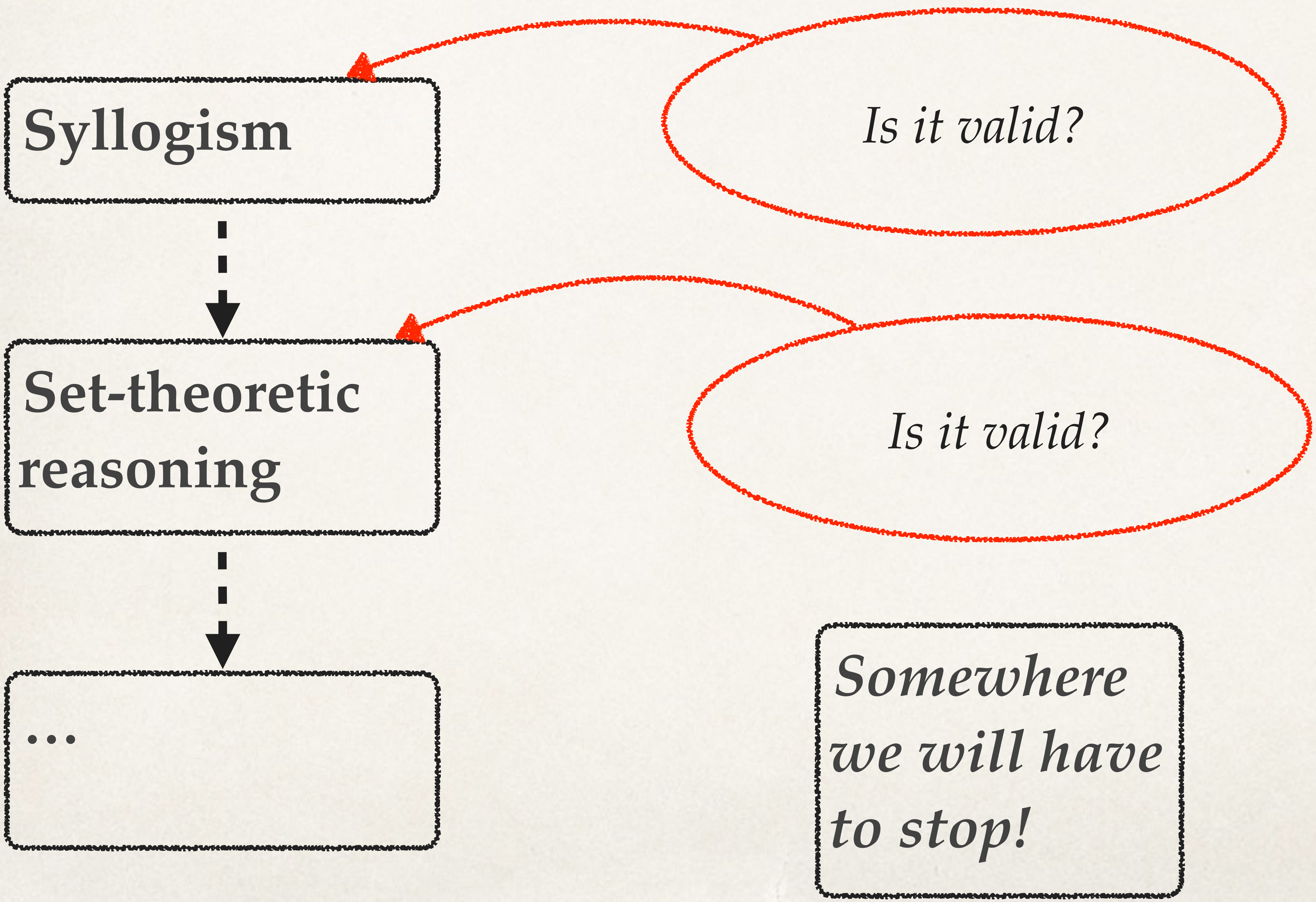
From Monday Class: Checking the Validity of a Syllogism

Syllogism	**Set-theoretic translation**
All A are B **All** B are C ——————— **All** A are C	$\boldsymbol{A \subseteq B}$ $\boldsymbol{B \subseteq C}$ ——————— $\boldsymbol{A \subseteq C}$

We showed the validity of the syllogism by **relying on reasoning about** $\subseteq$

But how do we know that the reasoning about the subset relation $\subseteq$ *is itself valid?*

From Monday: An Infinite Regress



Can We Take Reasoning About Sets for Granted? We cannot…

If we are not careful enough, our theory of sets generates a contradiction — this is Russell's paradox.

Now, from the contradiction anything follows, so any piece of set theoretic reasoning would follow.

Since we are using set theoretic reasoning to check the validity of our syllogistic patterns, **a contradiction in our theory of sets means that all syllogistic patterns can be shown to be valid**.

That is an unacceptable consequence.

Russell’s Paradox in Plain English

Consider the set of all sets that are not elements of themselves.

> If the set of all sets that are not elements of themselves is itself an element of itself, then it is not a element of itself.
>
> If the set of all sets that are not elements of themselves is not an element of itself, then it is an element of itself.

Either way, we get a contradiction.

What to Do, Then?

Sets exist within a universe **U** which is itself a set and from which the elements of the new sets we are defining are taken.

Instead of simply writing

$$\mathbf{B=\{x \mid x \text{ is a banana}\}}$$

we should—*strictly speaking*—write:

$$\mathbf{B=\{x \in U \mid x \text{ is a banana}\}}$$

Taming The Set of All Sets That Are Not Elements of Themselves

Consider the set of all sets that are not elements of themselves. Initially, we defined this set as follows:

$$\mathbf{R=\{x \mid x \notin x\}}.$$

But the set in question should be more properly defined as:

$$\mathbf{R^*=\{x \in U \mid x \notin x\}}.$$

Here the universe **U** is already a set from which we select the elements of our new set **R***.

How Does Having $R^*=\{x \in U \mid x \notin x\}$ as Opposed to $R=\{x \mid x \notin x\}$ Solve Russell's Paradox?

Given $\mathbf{R=\{x \mid x \notin x\}}$, we have

$$\mathbf{x \in R} \quad \textbf{iff} \quad \mathbf{x \notin x}$$

And by replacing **x** with **R**, we have

$$\mathbf{R \in R} \quad \textbf{iff} \quad \mathbf{R \notin R} \qquad \textbf{Contradiction!}$$

Given $\mathbf{R^*=\{x \in U \mid x \notin x\}}$, we have:

$$\mathbf{x \in R^*} \quad \textbf{iff} \quad \mathbf{x \in U} \text{ and } \mathbf{x \notin x}.$$

And by replacing **x** with $\mathbf{R^*}$, we have:

$$\mathbf{R^* \in R^*} \quad \textbf{iff} \quad \mathbf{R^* \in U} \text{ and } \mathbf{R^* \notin R^*}.$$

Here we get a contradiction provided $\mathbf{R^* \in U}$. To avoid the contradiction, we should assume $\mathbf{R^* \notin U}$.

If $\mathbf{R^* \notin U}$, no contradiction!

The Upshot of All This…

We can avoid the contradiction by assuming that $\mathbf{R^*} \notin \mathbf{U}$, that is, by assuming that the universe **U** does not contain any set we can possibly conceive.

So, to avoid the contradiction **we need to assume that the set of all sets does not exist**.

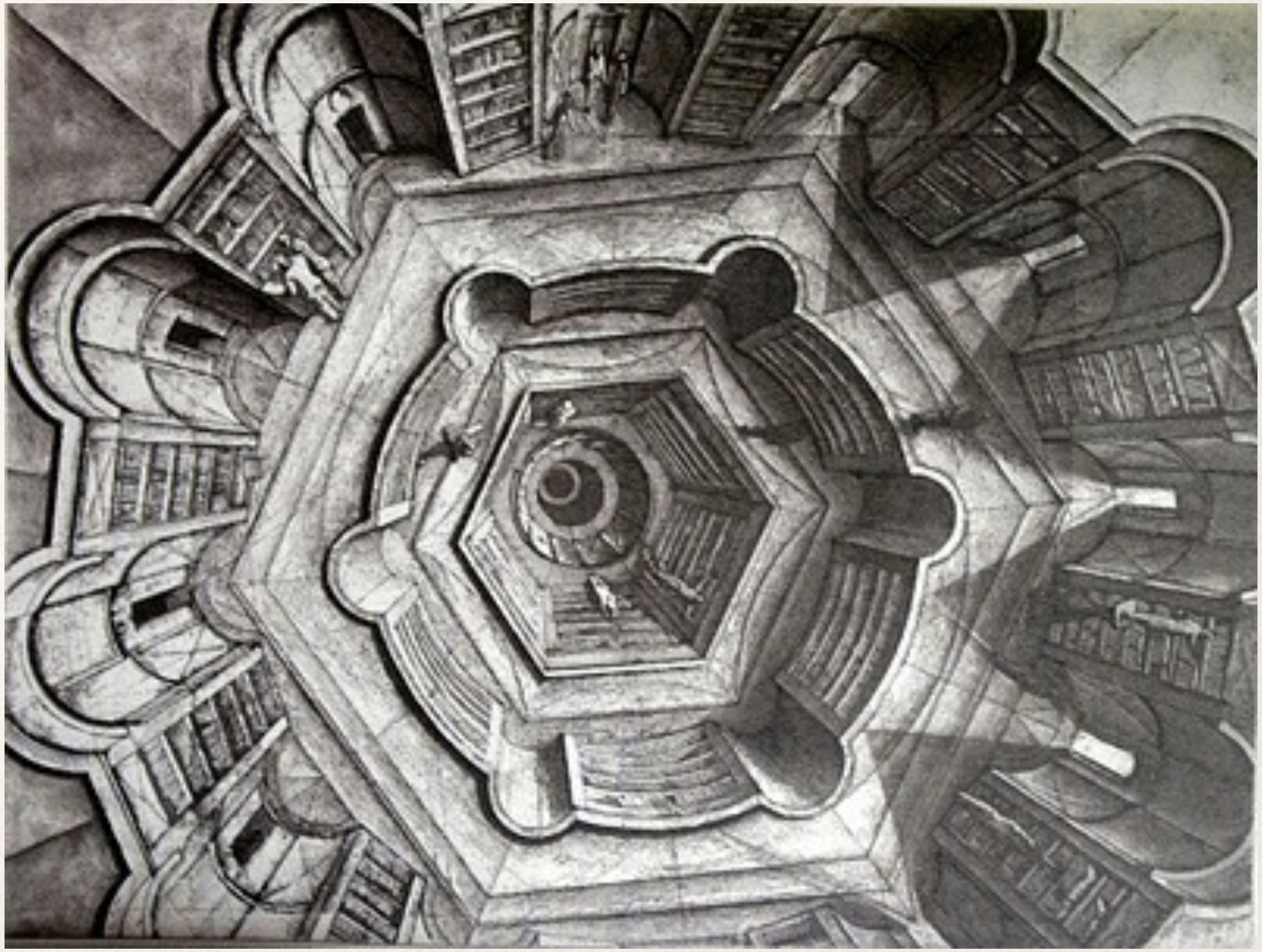
In somewhat more evocative terms, the upshot is that, on pain of contradiction, **set theory does not admit of objects (sets) that include the totality of reality**.

An Aside…
The Library of Babylon by Luis Borges

It is possible to conceive of a library that contains all knowledge possible?

Can there be a catalogue of all catalogues?

Can there be a catalogue of all catalogues that do not contain themselves as items in the catalogue?



Back to Syllogistic Validity or Invalidity

Are These Syllogistic Patterns Valid or Invalid?

NB: These are just 3 among the 64 total syllogistic patterns.

All A are B
All B are C

All A are C

No A is B
All C are A

No C is B

All A are B
Some C are **not** B

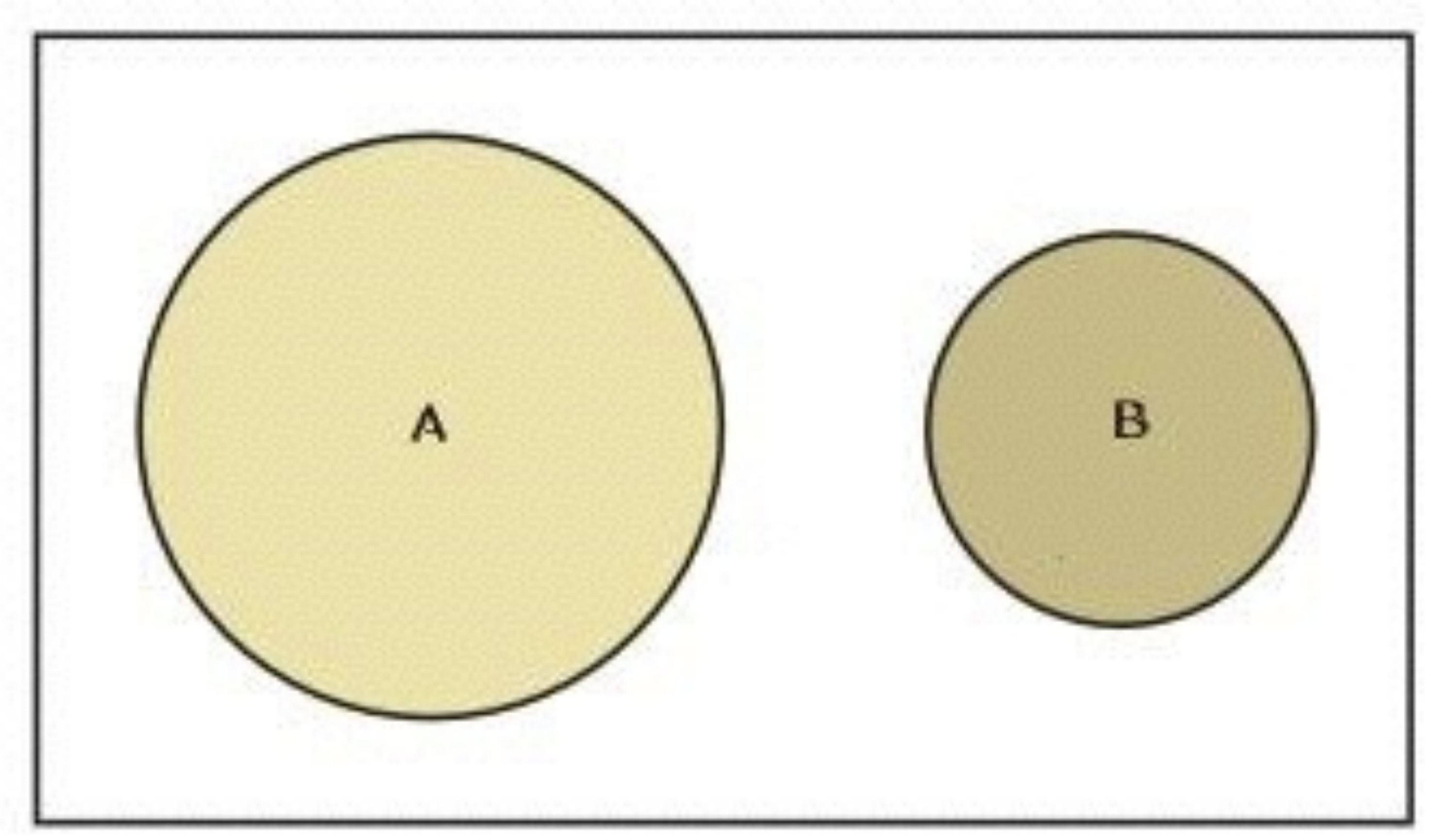
No C is A

Yes, insofar as the subset relation $\subseteq$ is transitive.

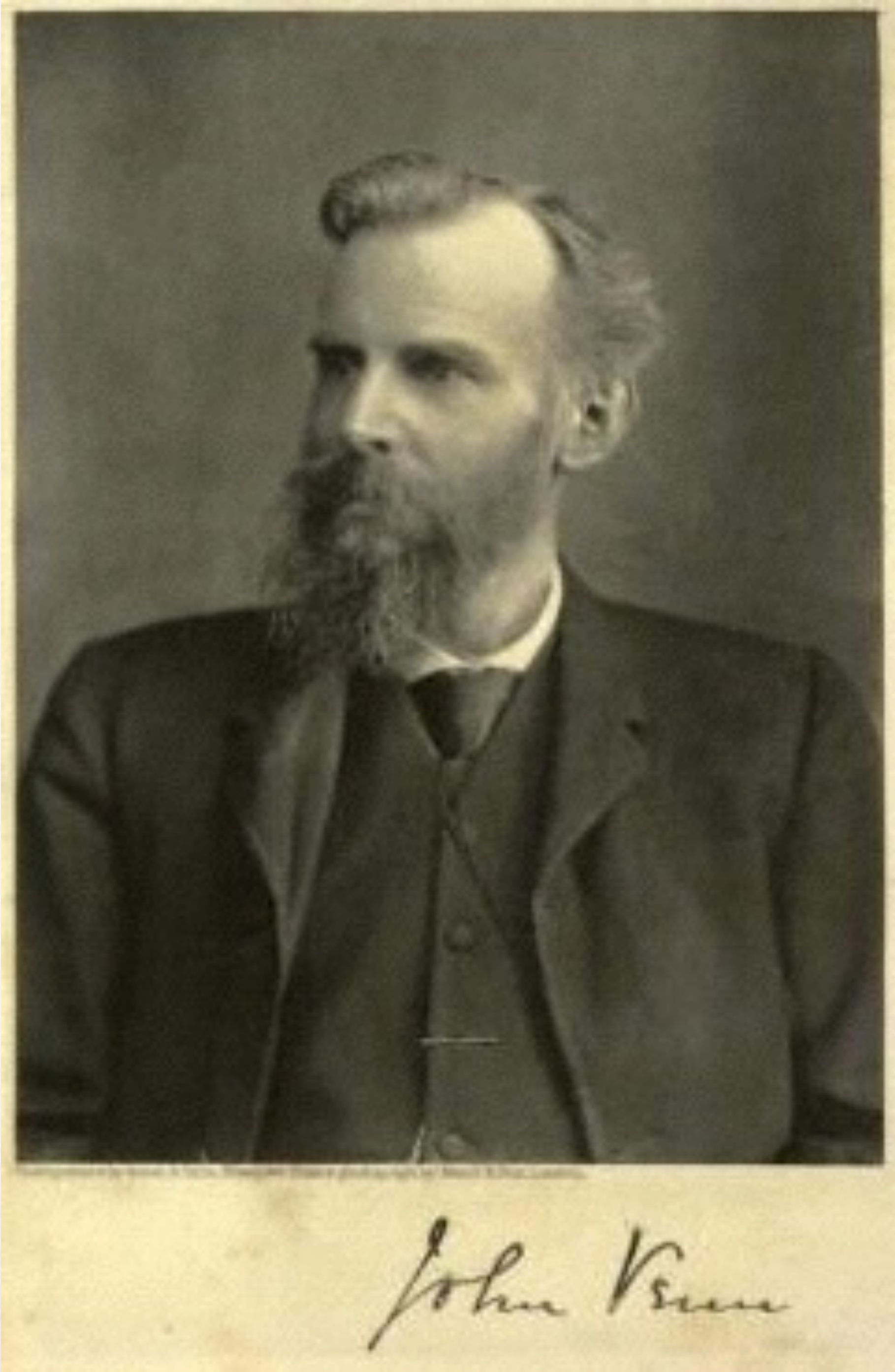
What about these other patterns? **We need to examine operations on sets**

Operation On Sets

Representation of Sets **A** and **B** as Diagrams



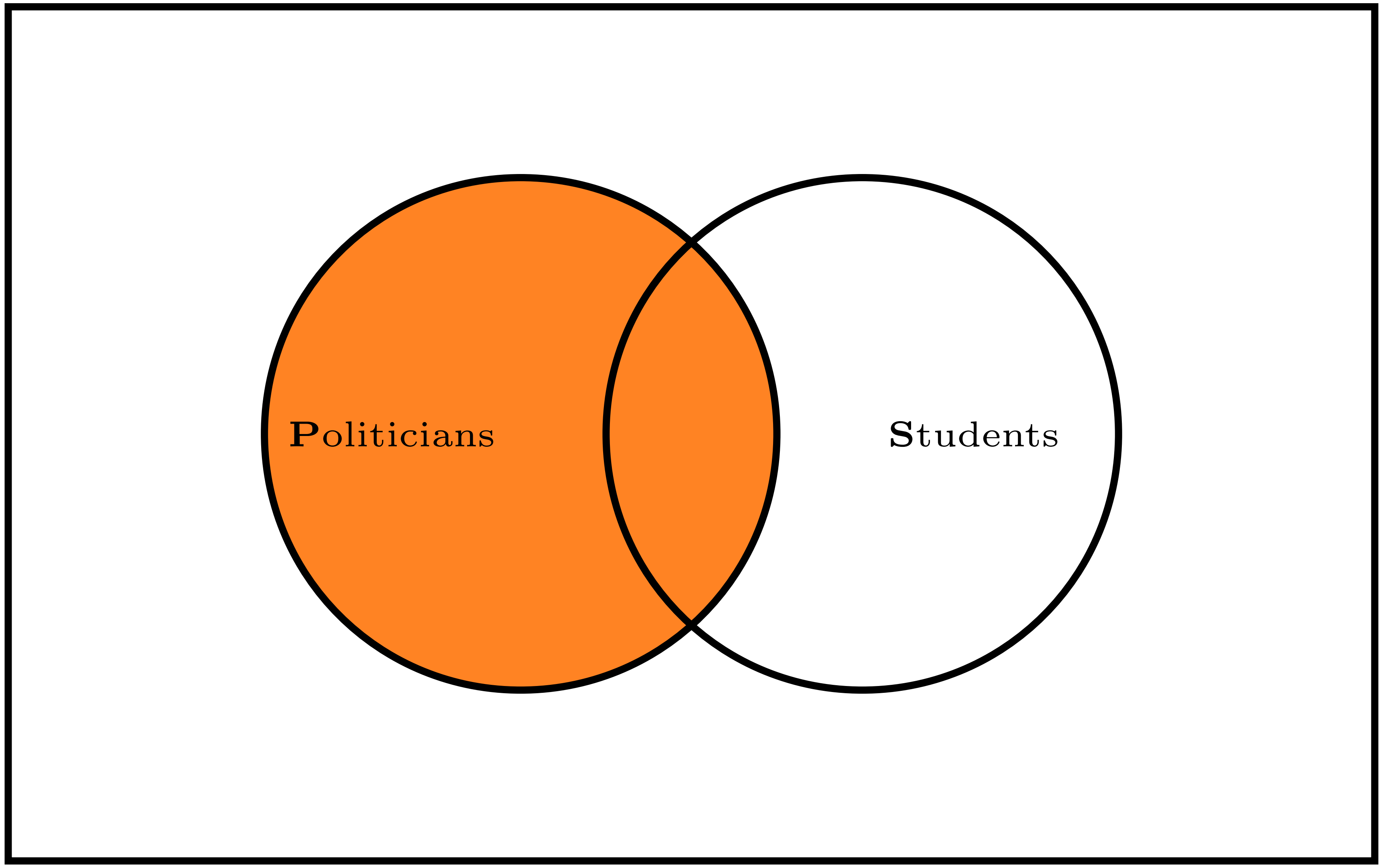
The idea of representing sets as colored regions is due to John Venn (1834-1923). Venn is known for the Venn's diagrams.



Sets as Diagrams (1)

The rectangle's area represents our universe U.

A **set**: **P**oliticians

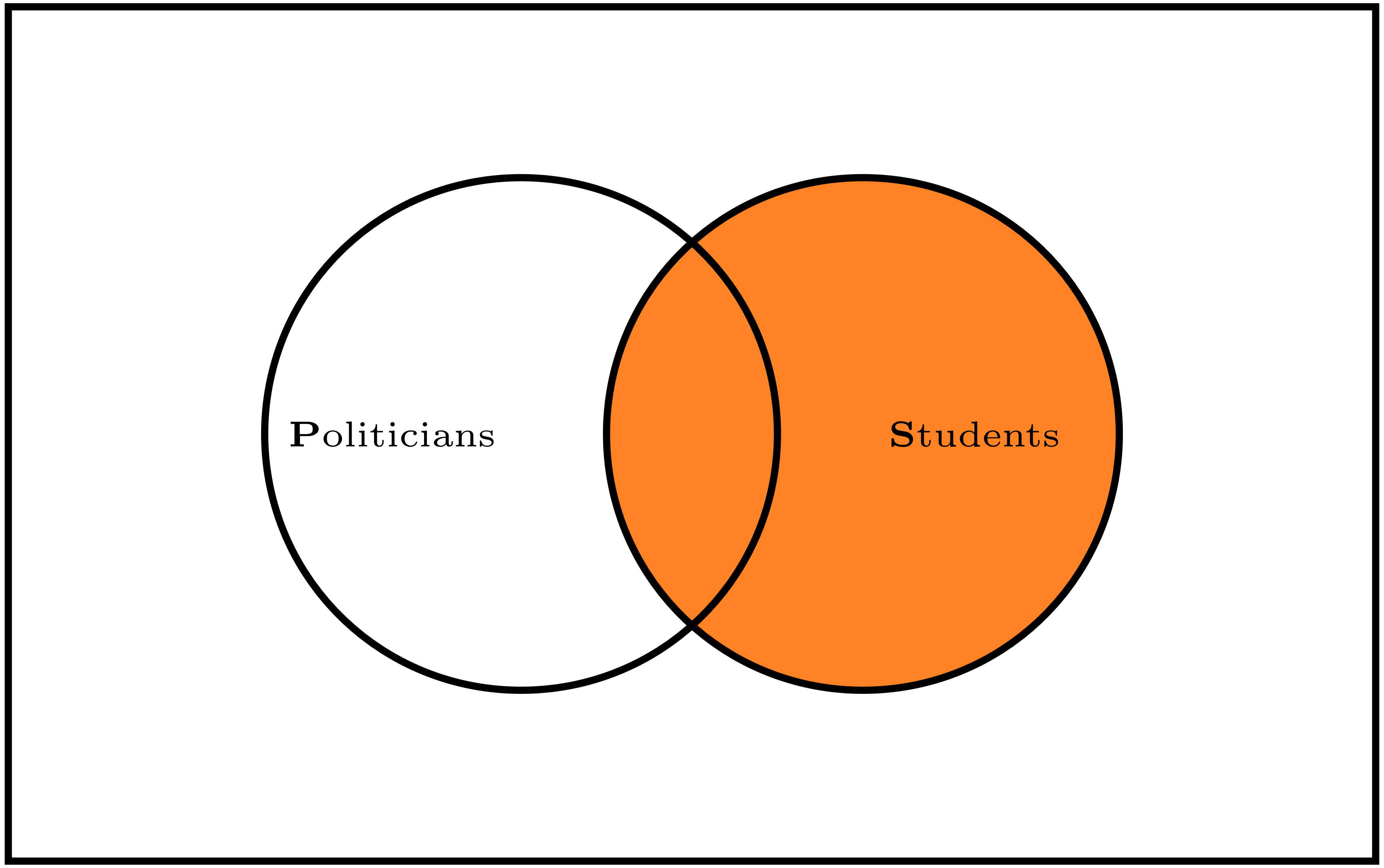


The orange colored area represents the set P of politicians

$\boldsymbol{P}$

Sets as Diagrams (2)

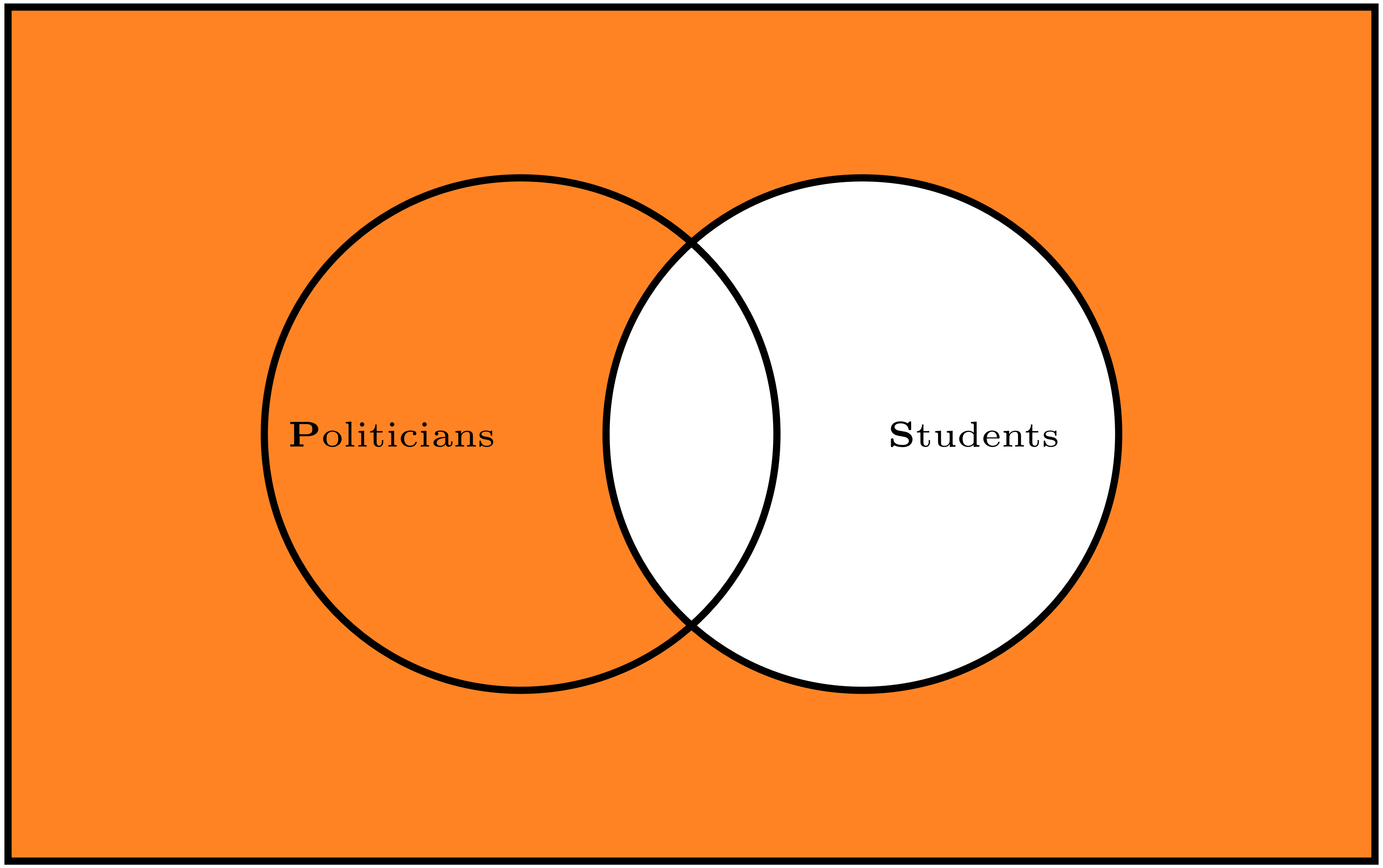
A **set**: **S**tudents



S

Complement of a Set (1)

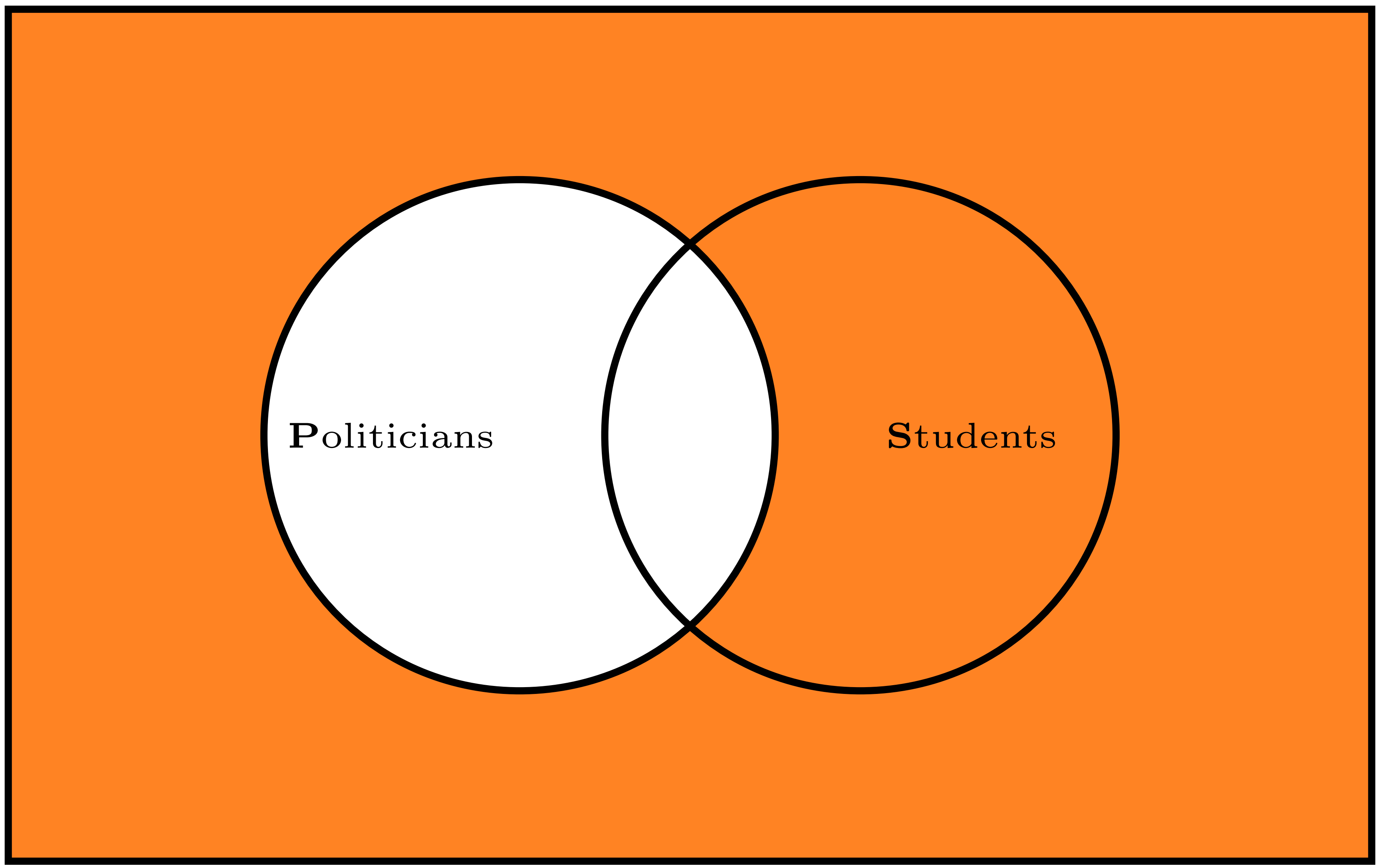
Complement: **No** **S**tudents



$\overline{S}$

Complement of a Set (2)

<u>Complement</u>: **No** **P**oliticians



$$\overline{P}$$

Complement and Negation

The set generated by the complement operation can be defined, as follows:

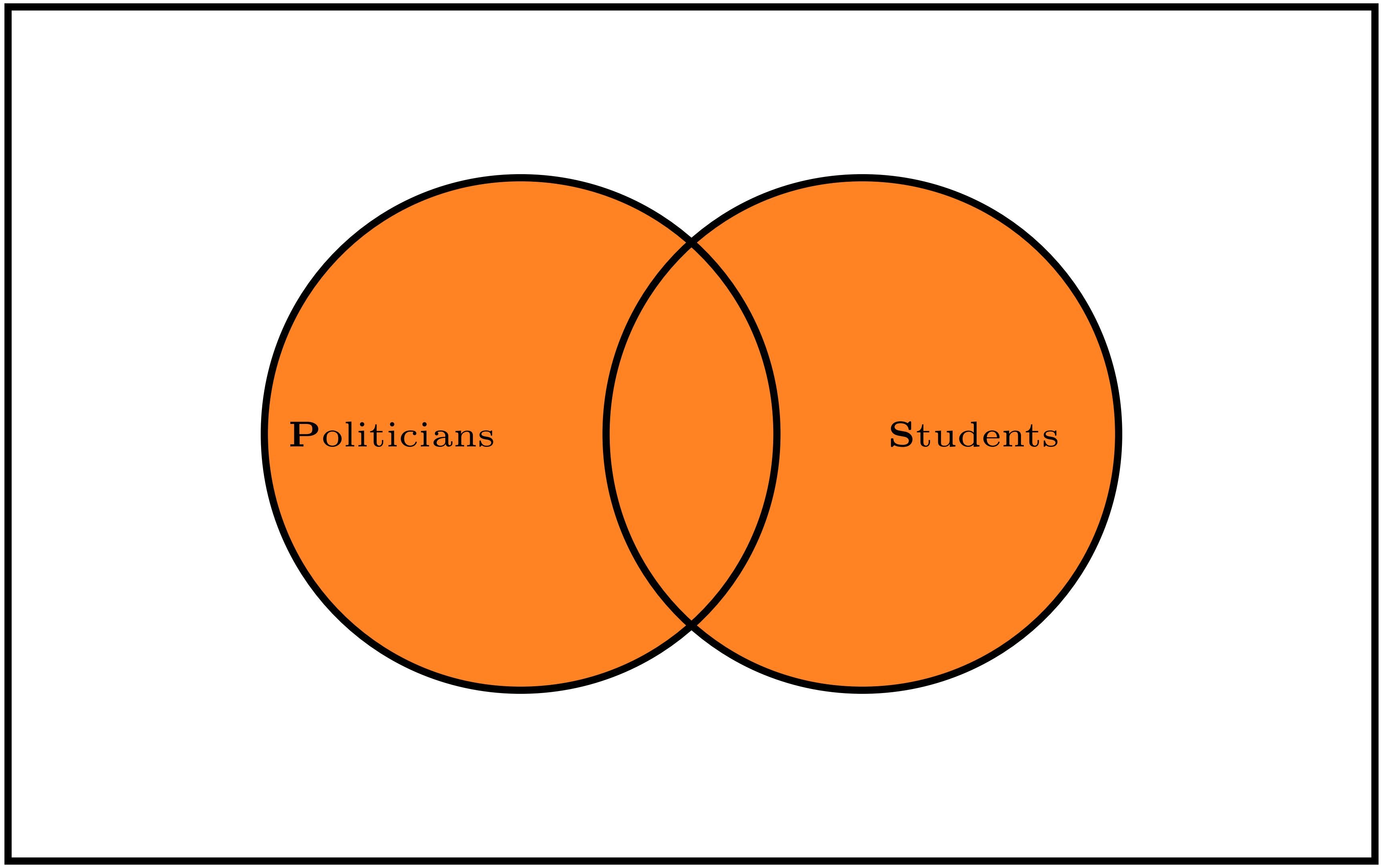
$$\mathbf{-A=\{\, x \mid \text{it is } \underline{\text{not}} \text{ the case that } x \in A\}}$$
$$\mathbf{=\{\, x \mid x \notin A\}}$$

Similarly, for a given element **a**, we say that

$\mathbf{a \in -A}$ **iff** $\mathbf{a \notin A}$

Union of Sets (1)

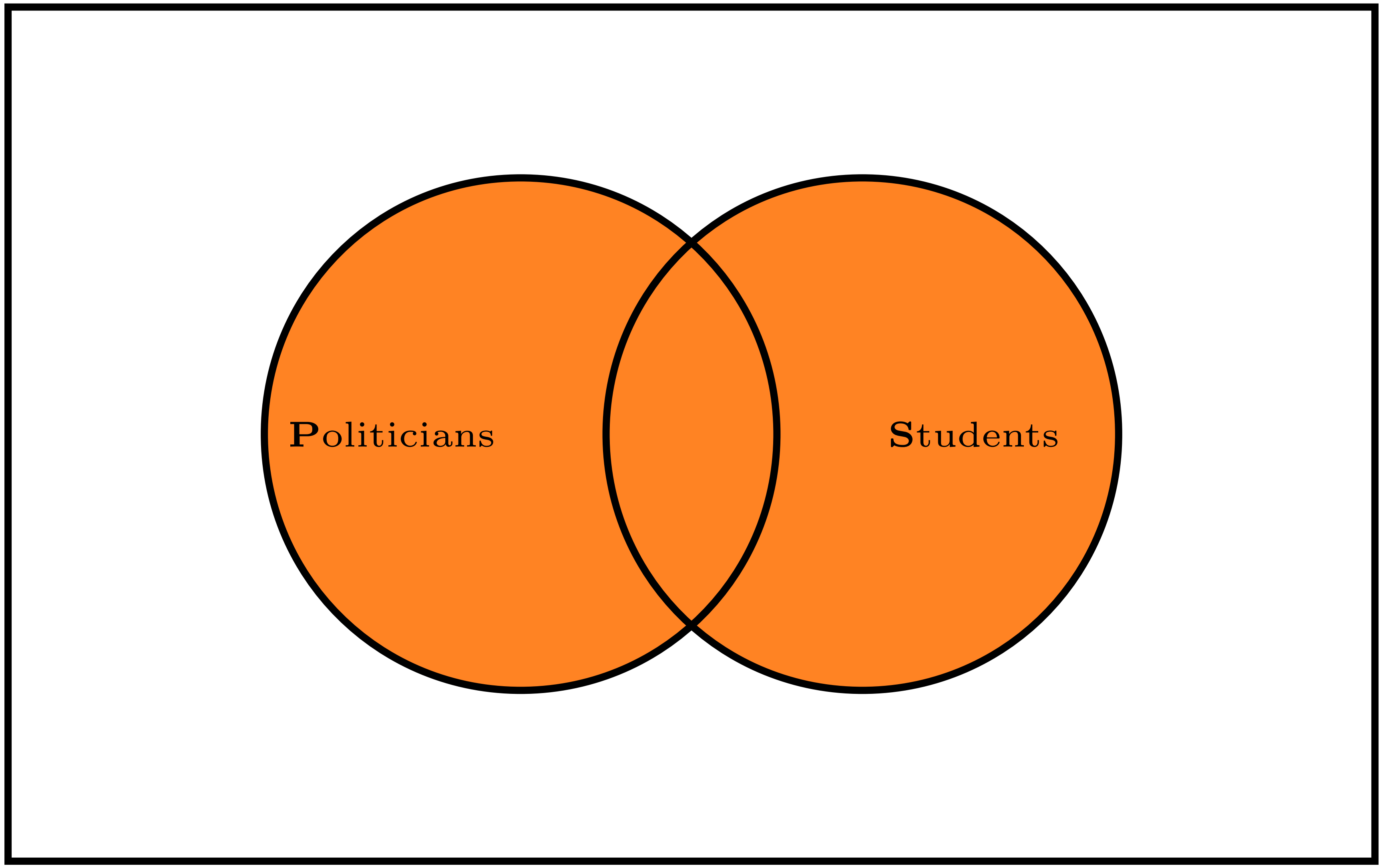
<u>Union</u>: **P**olitícians **or** **S**tudents



$$P \cup S$$

Union of Sets (2)

Union: **S**tudents **or** **P**oliticians



$$S \cup P$$

Union and Disjunction

The set generated by the union operation can be defined, as follows:

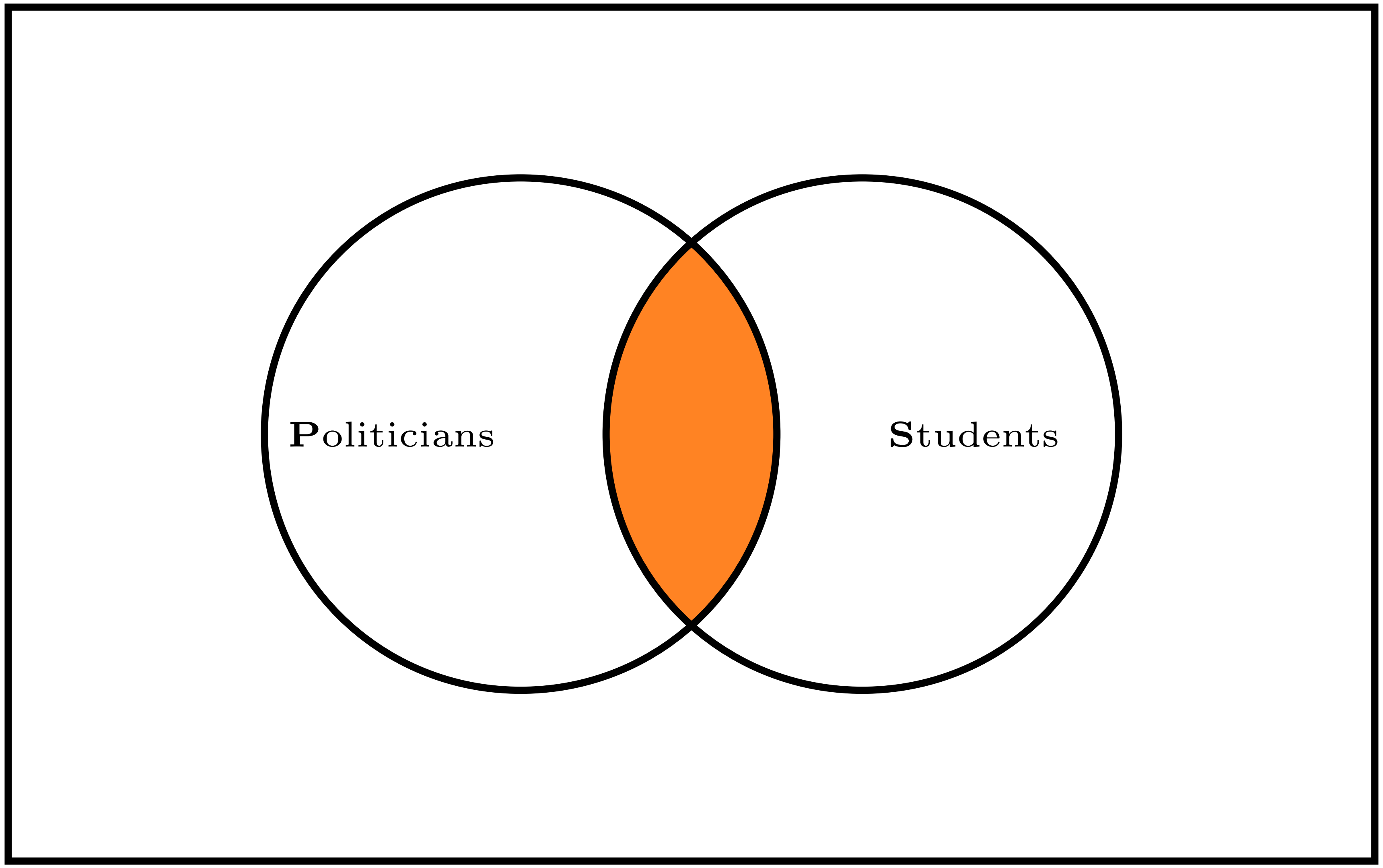
$$\mathbf{A} \cup \mathbf{B} = \{\, \mathbf{x} \mid \mathbf{x} \in \mathbf{A} \ \underline{\mathbf{or}}\ \mathbf{x} \in \mathbf{B} \}$$

Similarly, for a given element **a**, we say that

$\mathbf{a} \in \mathbf{A}$ <u>or</u> $\mathbf{a} \in \mathbf{B}$ **iff** $\mathbf{a} \in \boldsymbol{A} \cup \boldsymbol{B}$

Intersection of Sets (1)

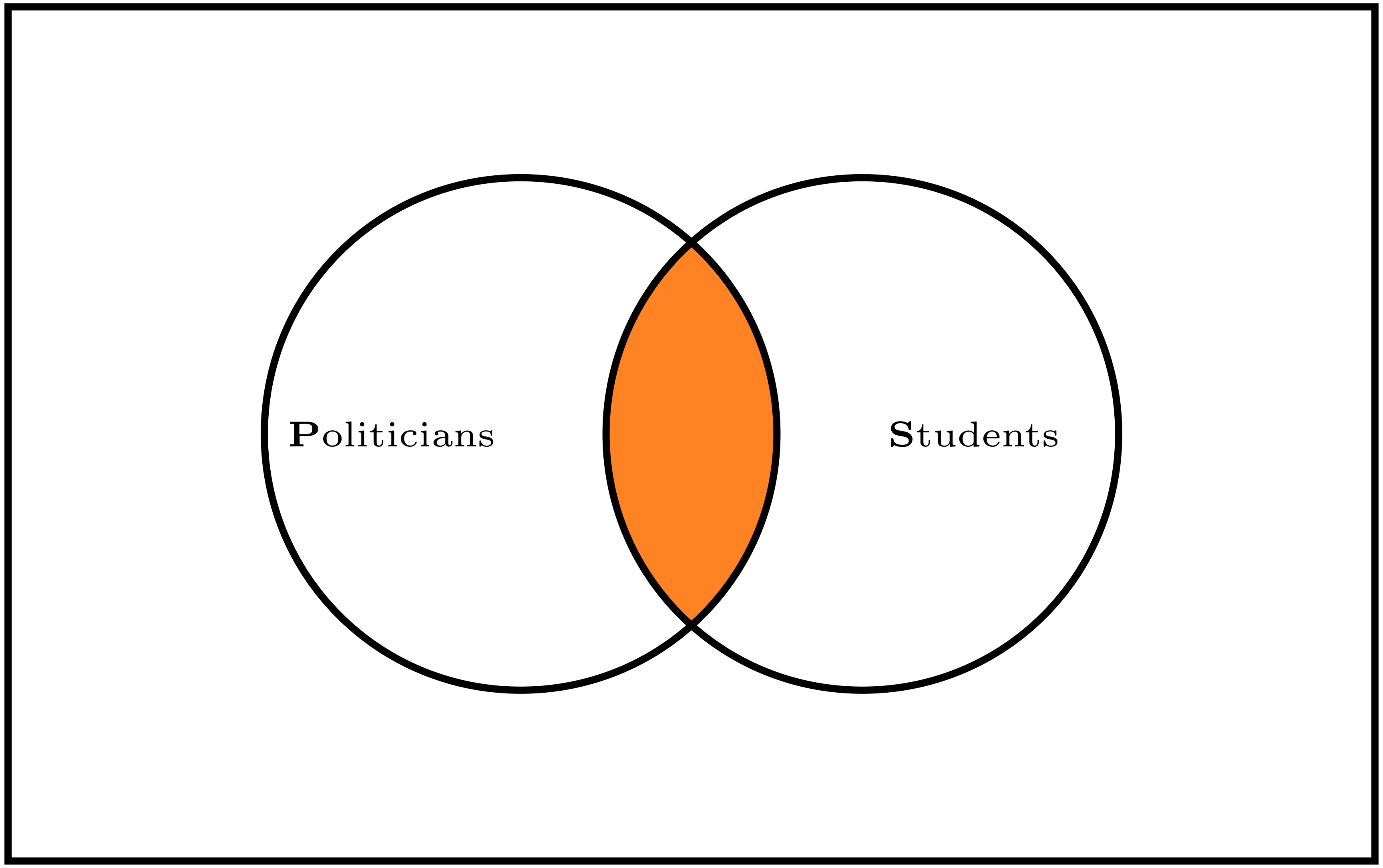
<u>Intersection</u>: **P**oliticians **and** **S**tudents



$$P \cap S$$

Intersection of Sets (2)

<u>Intersection</u>: **S**tudents **and** **P**oliticians



$$S \cap P$$

Intersection and Conjunction

The set generated by the intersection operation can be defined, as follows:

$$\mathbf{A \cap B = \{\, x \mid x \in A \ \underline{and}\ x \in B \}}$$

Similarly, for a given element **a**, we say that

a ∈ **A** <u>and</u> **a** ∈ **B** **iff** **a** ∈ $\boldsymbol{A \cap B}$

We Can Now Check the Validity of All Syllogistic Patterns

Checking Validity

Syllogism

No *A* is *B*
All *C* are *A*

No *C* is *B*

Set-theoretic translation

$A \cap B = \varnothing$
$C \subseteq A$

$C \cap B = \varnothing$

Is the Syllogism valid?

We need to check **whether $C \cap B = \varnothing$ follows from $A \cap B = \varnothing$ and $C \subseteq A$.** If it does, we can say that the syllogism in question is valid.

Suppose $\boldsymbol{A} \cap \boldsymbol{B} = \varnothing$ and $\boldsymbol{C} \subseteq \boldsymbol{A}$. We should prove that $\boldsymbol{C} \cap \boldsymbol{B} = \varnothing$.

Suppose (*) $\boldsymbol{A} \cap \boldsymbol{B} = \varnothing$ and (**) $\boldsymbol{C} \subseteq \boldsymbol{A}$.

In order to establish that $\boldsymbol{C} \cap \boldsymbol{B} = \varnothing$, we need to show that no element belongs to $\boldsymbol{C} \cap \boldsymbol{B}$.

Suppose for contradiction that there is an element $\mathbf{a}$ such that $\mathbf{a} \in \boldsymbol{C} \cap \boldsymbol{B}$. This means that $\mathbf{a} \in \mathbf{C}$ <u>and</u> $\mathbf{a} \in \mathbf{B}$.

By assumption (**), we have that $\mathbf{C} \subseteq \mathbf{A}$, so $\mathbf{a} \in \mathbf{A}$.

So, we have that $\mathbf{a} \in \mathbf{A}$ and $\mathbf{a} \in \mathbf{B}$, whence $\mathbf{a} \in A \cap B$.

So, there is an $\mathbf{a} \in \mathbf{A} \cap \mathbf{B}$, whence $A \cap B \neq \varnothing$ which contradicts (*). So, there is no element $\mathbf{a}$ such that $\mathbf{a} \in \boldsymbol{C} \cap \boldsymbol{B}$.

And now a step-by-step proof of the same claim….

Suppose $A \cap B = \varnothing$ and $C \subseteq A$. We should prove that $C \cap B = \varnothing$.

Suppose (*) $A \cap B = \varnothing$ and (**) $C \subseteq A$.

In order to establish that $C \cap B = \varnothing$, we need to show that no element belongs to $C \cap B$.

Suppose for contradiction that (***) there is an element **a** such that **a** $\in C \cap B$.

1. From (***), $\mathbf{a} \in \mathbf{C}$ and $\mathbf{a} \in \mathbf{B}$.
2. So, $\mathbf{a} \in \mathbf{C}$.
3. From (**), $\mathbf{C} \subseteq \mathbf{A}$, so $\mathbf{a} \in \mathbf{A}$ from 2.
4. So, $\mathbf{a} \in \mathbf{A}$ and $\mathbf{a} \in \mathbf{B}$ from 3 and 1.
5. So, $\mathbf{a} \in A \cap B$.
6. So, $A \cap B \neq \varnothing$ and this contradicts (*).

Checking Invalidity

Syllogism

All A are B
Some C are **not** B

No C is A

Set-theoretic translation

$A \subseteq B$
$C \nsubseteq B$

$C \cap A = \varnothing$

Is the Syllogism valid?

We need to check **whether** $C \cap A = \varnothing$ **follows from** $C \nsubseteq B$ **and** $A \subseteq B$. ***In fact, it does not follow, so the syllogism is invalid.*** *(next page)*

Counterexample to Validity

All tomatoes are rotten
Some chickpeas are **not** rotten

No chickpeas are tomatoes

All A is B
Some C are **not** B

No C is A

$A \subseteq B$
$C \nsubseteq B$

$C \cap A = \varnothing$

Counterexample:

Tomatoes = {a}
Rotten = {a, b}
Chickpeas = {a, b, c}

Counterexample:

A = {a}
B = {a, b}
C = {a, b, c}

Suppose $A \subseteq B$ and $C \nsubseteq B$. Does it follow that $C \cap \mathbf{A} = \varnothing$?

We construct a counterexample. Let

$\mathbf{A} = \{\mathbf{a}\}$
$\mathbf{B} = \{\mathbf{a}, \mathbf{b}\}$
$\mathbf{C} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

Note that both $A \subseteq B$ and $C \nsubseteq B$ are satisfied.

By construction, $\mathbf{a} \in \mathbf{C}$ and $\mathbf{a} \in \mathbf{A}$, so $\mathbf{a} \in C \cap \mathbf{A}$, whence $C \cap \mathbf{A} \neq \varnothing$.

Even if $A \subseteq B$ and $C \nsubseteq B$, it doesn't follow that $C \cap \mathbf{A} = \varnothing$.

On Counterexamples Involving Sets

Whenever you construct a counterexample, define your sets by defining which elements belong to them.

Use **very simple sets with as few elements as possible**.

If possible, avoid complicated counterexamples.

You need to construct **one (simple) counterexample only** to show that a syllogism is invalid.