Object language	Meta-language
$\rightarrow$	implies
$\neg$	not
$\wedge$	and
$\vee$	or
$\forall$	for all
$\exists$	there exists

Object language *Meta-language*

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PHIL 50 - Introduction to Logic

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Week 7 — Friday Class - Syntax and Semantics of Predicate Logic

Summary

Linguistic ingredient	*Function **I** or **g***	*Value of **I** or **g***
constant symbol c	$I(c)$	one object
1-place predicate symbol P	$I(P)$	set of objects
2-place predicate symbol R	$I(R)$	set of ordered pairs of objects

variable x	$g(x)$	one object
variable x	$g_{[x:=d]}(x)$	object d

Truth Conditions for Quantified Formulas

Truth condition for existentially quantified formulas

$\langle \boldsymbol{D, I, g} \rangle \models \exists \mathbf{x} \phi$ *iff* there is $\boldsymbol{d}$ [$\boldsymbol{d} \in \boldsymbol{D}$ and $\langle \boldsymbol{D, I, g_{[x:=d]}} \rangle \models \phi$]

Truth condition for universally quantified formulas

$\langle \boldsymbol{D, I, g} \rangle \models \forall \mathbf{x} \phi$ *iff* for all $\mathbf{d}$ [*if* $\boldsymbol{d} \in \boldsymbol{D}$, *then* $\langle \boldsymbol{D, I, g_{[x:=d]}} \rangle \models \phi$]

Correct, but Somewhat Lazy Formulations

$\langle \boldsymbol{D, I, g} \rangle \vDash \exists \mathbf{x}\phi$ *iff* there is a $\boldsymbol{d} \in \boldsymbol{D}$ such that $\langle \boldsymbol{D, I, g_{[x:=d]}} \rangle \vDash \phi$

$\langle \boldsymbol{D, I, g} \rangle \vDash \exists \mathbf{x}\phi$ *iff* there is a $\boldsymbol{d} \in \boldsymbol{D}$ and $\langle \boldsymbol{D, I, g_{[x:=d]}} \rangle \vDash \phi$

$\langle \boldsymbol{D, I, g} \rangle \vDash \forall \mathbf{x}\phi$ *iff* for all $\boldsymbol{d} \in \boldsymbol{D}$, $\langle \boldsymbol{D, I, g_{[x:=d]}} \rangle \vDash \phi$

You can use the above formulations for $\exists$ and $\forall$ but be wary that what is actually meant is:

$\langle \boldsymbol{D, I, g} \rangle \vDash \exists \mathbf{x}\phi$ *iff* there is $\boldsymbol{d}$ [$\boldsymbol{d} \in \boldsymbol{D}$ and $\langle \boldsymbol{D, I, g_{[x:=d]}} \rangle \vDash \phi$]

$\langle \boldsymbol{D, I, g} \rangle \vDash \forall \mathbf{x}\phi$ *iff* for all **d** [*if* $\boldsymbol{d} \in \boldsymbol{D}$, *then* $\langle \boldsymbol{D, I, g_{[x:=d]}} \rangle \vDash \phi$]

Let's now Look at Formulas with the **Propositional Connectives**

Truth Conditions for Formulas Containing the Connectives

The connectives in predicate logic do not behave any differently from propositional logic. However, the way in which we shall write their truth conditions is slightly different from what we did in the case of propositional logic.

$\mathbf{M} \models \neg \phi$	*iff*	*it is not the case that* $\mathbf{M} \models \phi$, *i.e.* $\mathbf{M} \not\models \phi$
$\mathbf{M} \models \phi \wedge \psi$	*iff*	$\mathbf{M} \models \phi$ *and* $\mathbf{M} \models \psi$
$\mathbf{M} \models \phi \vee \psi$	*iff*	$\mathbf{M} \models \phi$ *or* $\mathbf{M} \models \psi$
$\mathbf{M} \models \phi \rightarrow \psi$	*iff*	$\mathbf{M} \models \phi$ *implies* $\mathbf{M} \models \psi$

Assessing the Truth of Formulas with Constants, Predicate Symbols, and Connectives

$I(a) =$  $I(b) =$  $I(c) =$ 

$I(A) = \{$  ,  $\}$ $I(B) = \{$  ,  $\}$ $I(C) = \{$  ,  $\}$

$I(Eat) = \{ \langle$  ,  $\rangle , \langle$  ,  $\rangle \}$

$\mathbf{M} \models \neg\, \mathbf{A(c)}$

$\mathbf{M} \models \mathbf{Eat}(c, a) \wedge \mathbf{Eat}(c, b)$

$\mathbf{M} \models \mathbf{Eat}(a, c) \rightarrow \mathbf{Eat}(c, b)$

b/c $\mathbf{I(c)} \notin I(A)$

b/c $\langle \mathbf{I(c),I(a)} \rangle \in \mathbf{I(Eat)}$ *and* $\langle \mathbf{I(c),I(b)} \rangle \in \mathbf{I(Eat)}$

b/c $\langle \mathbf{I(a),I(c)} \rangle \in \mathbf{I(Eat)}$ *implies* $\langle \mathbf{I(c), I(b)} \rangle \in \mathbf{I(Eat)}$

[vacuously b/c antecedent is false]

Summary: Truth Conditions for Formulas in Predicate Logic so far

$\langle D, I, g\rangle \vDash P(c)$ *iff* $I(c) \in I(P)$

$\langle D, I, g\rangle \vDash R(c_1, c_2)$ *iff* $\langle I(c_1), (I(c_2) \rangle \in I(R)$

$\langle D, I, g\rangle \vDash \neg \phi$ *iff* $\langle D, I, g\rangle \nvDash \phi$

$\langle D, I, g\rangle \vDash \phi \wedge \psi$ *iff* $\langle D, I, g\rangle \vDash \phi$ *and* $\langle D, I, g\rangle \vDash \psi$

$\langle D, I, g\rangle \vDash \phi \vee \psi$ *iff* $\langle D, I, g\rangle \vDash \phi$ *or* $\langle D, I, g\rangle \vDash \psi$

$\langle D, I, g\rangle \vDash \phi \rightarrow \psi$ *iff* $\langle D, I, g\rangle \vDash \phi$ *implies* $\langle D, I, g\rangle \vDash \psi$

$\langle D, I, g\rangle \vDash \exists x\phi$ *iff* there is d [$d \in D$ and $\langle D, I, g_{[x:=d]}\rangle \vDash \phi$]

$\langle D, I, g\rangle \vDash \forall x\phi$ *iff* for all $\mathbf{d}$ [*if* $d \in D$, *then* $\langle D, I, g_{[x:=d]}\rangle \vDash \phi$]

Recall our two Philosophical Questions from Monday:

(Q1)
How is it that words and sentences have meaning?

(Q2) How is it that sentences (but not words alone) can be true or false? And more generally, what is truth?

How can THIS mean THAT?

"The river is flowing surrounded by trees"

A Simple Suggestion

"The river is flowing surrounded by trees"

The word "river" means RIVER-IN-REALITY

The word "trees" means TREES-IN-REALITY

The word "surrounded" means SURROUNDED-IN-REALITY

The word "flowing" means FLOWING-IN-REALITY

But is that all there is to the linguistic meaning of words and sentences?

What about the Meaning of "Is" and of the Logical Connectives?

"The river IS flowing"

"The river IS NOT flowing"

"The river is flowing AND the trees are surrounding it."

Etc.

It seems that words such as "is", "not", "and" etc. do not have any direct correspondence to things in reality. *What do they mean then*?

Have We Made Any Progress?

The Meaning of "Is" as $\in$

We proceeded in two steps:

1. translating a natural language sentences, e.g. "**John is tall**", into a formula of predicate logic, e.g. ***Tall(john)***;
2. assigning meaning and truth conditions to ***Tall(john)*** by means of
   - the interpretation function ***I***
   - set-theoretic membership relation $\in$

$\langle D, I, g \rangle \vDash Tall(john)$ *iff* $\mathbf{I(john)} \in \mathbf{I(Tall)}$

*The copula "**is**" is given a meaning by **<u>translating</u>** it into set-theoretic membership "$\in$"*

Very well, but what's the meaning of "$\in$" then?

The Meaning of the Connectives (1)

$M \vDash \neg\, \phi$	*iff*	***not*** $M \vDash \phi$ **, i.e.** $M \nvDash \phi$
$M \vDash \phi \wedge \psi$	*iff*	$M \vDash \phi$ ***and*** $M \vDash \psi$
$M \vDash \phi \vee \psi$	*iff*	$M \vDash \phi$ ***or*** $M \vDash \psi$
$M \vDash \phi \rightarrow \psi$	*iff*	$M \vDash \phi$ ***implies*** $M \vDash \psi$

The meaning of the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ *is given by the natural language connectives "**not**", "**and**", "**or**" and "**implies**"*

Isn't that completely circular and uninteresting? We have been given no account whatsoever of the meaning of the connectives!

The Meaning of the Connectives (2)

To avoid the circularity worry, we can work algebraically, saying e.g. that the meaning of ∧ is the function ***min*** and the meaning of ∨ is the function ***max***. We took this approach when we studied the connectives in propositional logic during week 2 of the course.

[¬ ϕ]=1	*iff*	**1-[ϕ]=1**
[ϕ ∧ ψ]=1	*iff*	***min*([ϕ], [ψ])=1**
[ϕ ∨ ψ]=1	*iff*	***max*([ϕ], [ψ])=1**
[ϕ→ ψ]=1	*iff*	***max*(1-[ϕ], [ψ])=1**

*The meaning of the logical connectives is given by **translating** them into mathematical functions*

But what's the meaning of "min", "max" etc?

Meaning and Translation

Sentences in natural language	*Formulas in predicate logic*	*Set-theoretic expressions*
Snow is white	$\mathbf{W(s)}$	$\mathbf{I(s)} \in \mathbf{I(W)}$
Snow is white and Erdos is old	$\mathbf{W(s)} \wedge \mathbf{O(e)}$	$\mathbf{I(s)} \in \mathbf{I(W)}$ and $\mathbf{I(e)} \in \mathbf{I(O)}$

Each step is a process of translation, i.e. from **natural language** to **predicate logic**, and from predicate logic to **math/set-theory**.
The underlying assumption is that the language of math / set-theory can offer us an insight into the meaning of natural language.

Taking Stock

(Q1)
How is it that words and sentences have meaning?

*We did not get a full answer to this question. Still, the suggestion is that we can grasp the meaning of some natural language expressions by translating them into set-theoretic and mathematical language, e.g. "IS" is translated as "$\in$" and "AND" is translated as "**min**".*

*The meaning of "$\in$" and "**min**" is taken for granted.*

The Second Question

(Q2) How is it that sentences (but not words alone) can be true or false? And more generally, what is truth?

Let’s Now Try to Understand How Truth Conditions are Written

The Distinction Between Object Language and Meta-language

The **object language** consists of the formulas of predicate logic, such as: $A(b)$, $A(b) \wedge B(c)$, $\forall x(A(x)$, etc.

The **meta-language** consists of the expressions that we use to talk about the object language, such as: "$\langle D, I, g\rangle \vDash$," **"and"**, **"iff"** etc.

The Distinction Between Object Language and Meta-language in the Truth Conditions

$\langle D, I, g\rangle \vDash \boxed{\mathbf{P}(c)}$ *iff* $\mathbf{I}(\boxed{\mathbf{c}}) \in \mathbf{I}(\boxed{\mathbf{P}})$

$\langle D, I, g\rangle \vDash \boxed{\neg\phi}$ *iff* ***not*** $\langle D, I, g\rangle \vDash \boxed{\phi}$, *i.e.* $\langle D, I, g\rangle \nvDash \boxed{\phi}$

$\langle D, I, g\rangle \vDash \boxed{\phi \wedge \psi}$ *iff* $\langle D, I, g\rangle \vDash \boxed{\phi}$ ***and*** $\langle D, I, g\rangle \vDash \boxed{\psi}$

$\langle D, I, g\rangle \vDash \boxed{\phi \vee \psi}$ *iff* $\langle D, I, g\rangle \vDash \boxed{\phi}$ ***or*** $\langle D, I, g\rangle \vDash \boxed{\psi}$

$\langle D, I, g\rangle \vDash \boxed{\phi \rightarrow \psi}$ *iff* $\langle D, I, g\rangle \vDash \boxed{\phi}$ ***implies*** $\langle D, I, g\rangle \vDash \boxed{\psi}$

1. **P(c), ¬ϕ, ϕ∧ψ, ϕ∨ψ, ϕ→ψ** are place holders for formulas in the language of predicate logic. This is called the ***object language***.
2. Expressions such as "∈", "**not**", "**and**", "**or**", "**implies**" are not part of the language of predicate logic. Also, ***"⊨", "⟨D, I, g⟩", "iff"*** are not part of the language of predicate logic. They are part of the ***meta-language***.

The Parallelism Between Object Language and Meta-language (1)

$\langle D, I, g\rangle \vDash \neg \phi$ *iff* ***not*** $\langle D, I, g\rangle \vDash \phi$, *i.e.* $\langle D, I, g\rangle \nvDash \phi$

$\langle D, I, g\rangle \vDash \phi \wedge \psi$ *iff* $\langle D, I, g\rangle \vDash \phi$ ***and*** $\langle D, I, g\rangle \vDash \psi$

$\langle D, I, g\rangle \vDash \phi \vee \psi$ *iff* $\langle D, I, g\rangle \vDash \phi$ ***or*** $\langle D, I, g\rangle \vDash \psi$

$\langle D, I, g\rangle \vDash \phi \rightarrow \psi$ *iff* $\langle D, I, g\rangle \vDash \phi$ ***implies*** $\langle D, I, g\rangle \vDash \psi$

The connectives ¬, ∧, ∨, →
(which are part of the ***object language****)*
*have as counterparts the connectives "****not****", "****and****", "****or****" and "****implies****" (which are part of the* ***meta-language****)*

There is a certain degree of circularity here.

The Parallelism Between Object Language and Meta-language (2)

$\langle \boldsymbol{D, I, g} \rangle \vDash \exists \mathbf{x}\phi$ *iff* there is a $\boldsymbol{d} \in \boldsymbol{D}$ such that $\langle \boldsymbol{D, I, g_{[x:=d]}} \rangle \vDash \phi$

$\langle \boldsymbol{D, I, g} \rangle \vDash \forall \mathbf{x}\phi$ *iff* for all $\mathbf{d}$, *if* $\boldsymbol{d} \in \boldsymbol{D}$, *then* $\langle \boldsymbol{D, I, g_{[x:=d]}} \rangle \vDash \phi$

The quantifiers $\exists \boldsymbol{x}$ *and* $\forall \boldsymbol{x}$ *(which are part of the* ***object language****) have as counterparts the expressions "****there is****" and "****for all****" (which are part of the* ***meta-language****)*

Again, there is a certain degree of circularity here.

The Truth Conditions as a (Recursive) Definition of Truth (in a Model)

$\langle D, I, g \rangle \vDash P(c)$ *iff* $I(c) \in I(P)$

$\langle D, I, g \rangle \vDash R(c_1, c_2)$ *iff* $\langle I(c_1), (I(c_2) \rangle \in I(R)$

$\langle D, I, g \rangle \vDash \neg \phi$ *iff* $\langle D, I, g \rangle \nvDash \phi$

$\langle D, I, g \rangle \vDash \phi \wedge \psi$ *iff* $\langle D, I, g \rangle \vDash \phi$ *and* $\langle D, I, g \rangle \vDash \psi$

$\langle D, I, g \rangle \vDash \phi \vee \psi$ *iff* $\langle D, I, g \rangle \vDash \phi$ *or* $\langle D, I, g \rangle \vDash \psi$

$\langle D, I, g \rangle \vDash \phi \rightarrow \psi$ *iff* $\langle D, I, g \rangle \vDash \phi$ *implies* $\langle D, I, g \rangle \vDash \psi$

$\langle D, I, g \rangle \vDash \exists x \phi$ *iff* there is d [$d \in D$ and $\langle D, I, g_{[x:=d]} \rangle \vDash \phi$]

$\langle D, I, g \rangle \vDash \forall x \phi$ *iff* for all **d** [*if* $d \in D$, *then* $\langle D, I, g_{[x:=d]} \rangle \vDash \phi$]

The Second Question Addressed

(Q2) How is it that sentences (but not words alone) can be true or false? And more generally, what is truth?

Again, we did not get a complete answer to this question. Still, the truth conditions for formulas in predicate logic give us a precise and systematic way account of which formulas count as true (in a model) and which formulas count as false (in a model).

The truth conditions can be seen as a definition of truth.