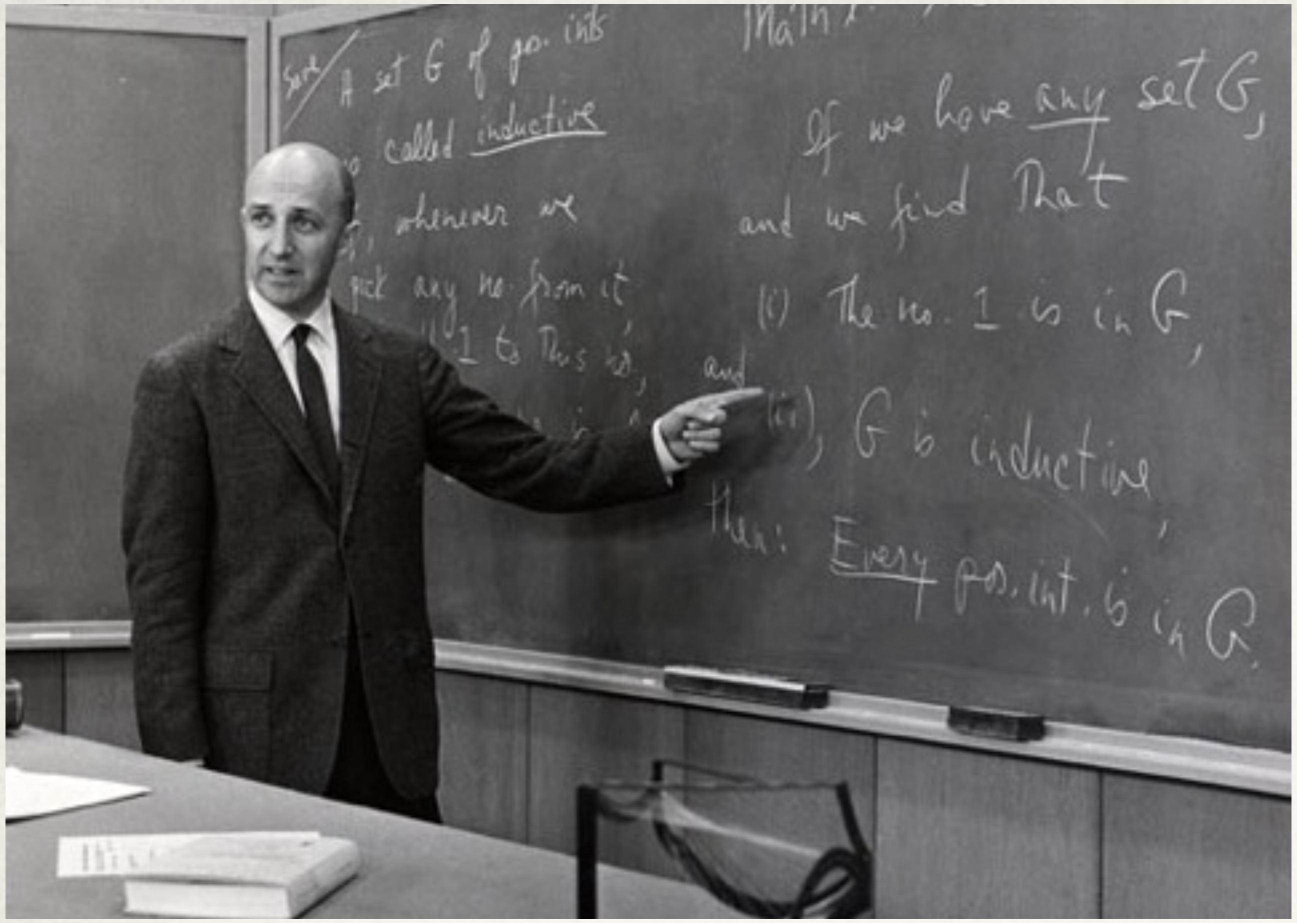


Leon Henkin

Kurt Gödel

PHIL 50 - Introduction to Logic

Marcello Di Bello, Stanford University, Spring 2014

Week 8 — Friday Class - Identity, Soundness and Completeness

Identity $=$

So far our language lacked a symbol for identity.
Let's now introduce a symbol for identity.

What Does "=" Mean?

$\langle D, I, g\rangle \vDash (c_1 = c_2)$ *iff* $\langle \mathbf{I(c_1), I(c_2)}\rangle \in \mathbf{I(=)}$

$\langle D, I, g\rangle \vDash (\mathbf{x} = y)$ *iff* $\langle \mathbf{g(x), g(y)}\rangle \in \mathbf{I(=)}$

I(=) is a set of pairs because "=" is a *two-place predicate* after all. What's peculiar about **I(=)** is that each pair in the set must consist of the same object twice.

Illustration:

$I(=) = \{\ \ldots$

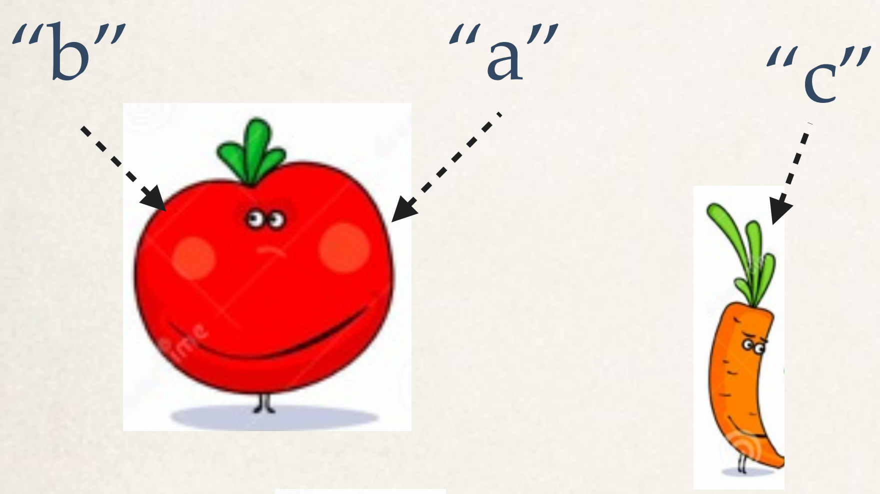
$\langle$  ,  $\rangle$, $\langle$  ,  $\rangle$

$\ldots\ \}$

Illustration

$M \vDash (c_1 = c_2)$ *iff* $\langle \mathbf{I(c_1), I(c_2)} \rangle \in \mathbf{I(=)}$

Let *M* be as follows:



I(b)= 

I(a)= 

I(c)= 

a=b is true in M because $\langle \mathbf{I(a), I(b)} \rangle \in \mathbf{I(=)}$

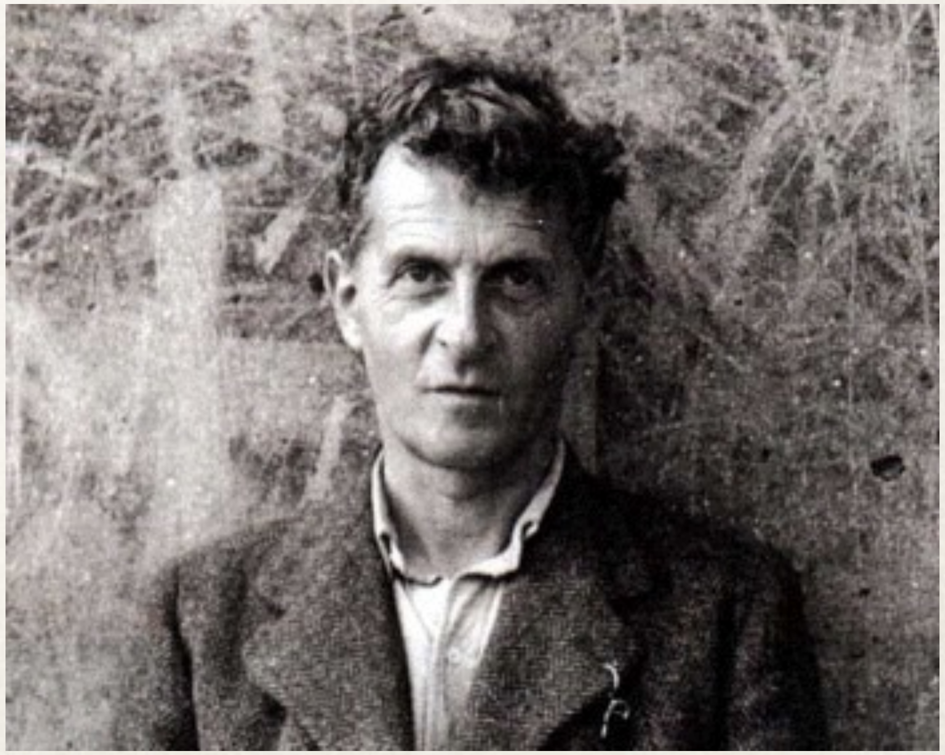
c=a is false in M because $\langle \mathbf{I(c), I(a)} \rangle \notin \mathbf{I(=)}$

¬(a=c) is true in M because $\langle \mathbf{I(a), I(c)} \rangle \notin \mathbf{I(=)}$

Isn't Identity Really Uninteresting?

"*Roughly speaking, to say of two things that they are identical is nonsense, and to say of one thing that it is identical with itself is to say nothing at all.*"

Ludwig Wittgenstein, *Tractatus* 5.5303



Identity Allows Us to Express Some Moderately Interesting Things

There Are at Least….

There are at least **two** objects

$\exists x \exists y \neg(x=y)$

There are at least **three** objects

$\exists x \exists y \exists z(\neg(x=y) \land \neg(x=z) \land \neg(y=z))$

There Are at Most….

There are at most **two** objects

$\forall x \forall y \forall z((x=y) \vee (x=z))$

There are at most **three** objects

$\forall x \forall y \forall z \forall s((x=y) \vee (x=z) \vee (x=s))$

There Are Exactly…

There are **exactly two** objects

There are at most **two** objects

$\forall x \forall y \forall z((x=y) \lor (x=z))$

There are at least **two** objects

$\exists x \exists y \neg(x=y)$

There are **exactly three** objects

There are at most **three** objects

$\forall x \forall y \forall z \forall s((x=y) \lor (x=z) \lor (x=s))$

There are at least **three** objects

$\exists x \exists y \exists z(\neg(x=y) \land \neg(x=z) \land \neg(y=z))$

All Is One!

$\forall x \forall y (x = y)$

Something is Everything!

$$\boxed{\exists x \forall y (x = y)}$$

Let's Now Return to Our Beloved Derivation Rules for Predicate Logic!

Recall (1): Derivation Rules for the Universal Quantifier

Conventions. (a) Let $\phi(x)$ be a placeholder for a formula of predicate logic of arbitrary complexity where x occurs free in ϕ. (b) Let $\phi(t)$ be the placeholder for a formula of predicate logic of arbitrary complexity, where t is a placeholder for a variable symbol or a constant symbol.

$$\frac{\forall x \phi(x)}{\phi(t)}\ \forall E$$

$$\frac{\phi(x)}{\forall x \phi(x)}\ \forall I$$

Restriction on ∀I

Variable x *cannot occur free in any uncanceled assumption on which* $\phi(x)$ *depends.*

Misapplication of ∀I

Let's say you know that
(1) x is a triangle;
(2) x is isosceles; and
(3) for all y, if y is isosceles,
then y has two equal sides.

From (2) and (3) it follows that
(4) x has two equal sides.
So, from (1) and (4), we have:
(5) if x is a triangle,
then x has two equal sides.
So, by universal introduction,
(6) for all x, if x is a triangle,
then x has two equal sides.

$$\dfrac{\dfrac{\dfrac{[I(x)]^2 \qquad \dfrac{\dfrac{[T(x)]^1}{[\forall y(I(y)\to E(y))]^3}}{I(x)\to E(x)}\,\forall E}{E(x)}\,\to E}{T(x)\to E(x)}\,\to I^1}{\forall x(T(x)\to E(x))}\,\forall I$$

The application of ∀I is wrong because ***x*** occurs free in the uncanceled assumption ***I(x)***

$$\text{INCORRECT!}\;\dfrac{\dfrac{\dfrac{[I(x)]^2 \qquad \dfrac{\dfrac{[T(x)]^1}{[\forall y(I(y)\to E(y))]^3}}{I(x)\to E(x)}\,\forall E}{E(x)}\,\to E}{T(x)\to E(x)}\,\to I^1}{\forall x(T(x)\to E(x))}\,\forall I$$

The application of ∀I is wrong because *x* occurs free in the uncanceled assumption ***I(x)***

$$\text{CORRECT!}\;\dfrac{\dfrac{\dfrac{\dfrac{[I(x)]^2 \qquad \dfrac{\dfrac{[T(x)]^1}{[\forall y(I(y)\to E(y))]^3}}{I(x)\to E(x)}\,\forall E}{E(x)}\,\to E}{T(x)\to E(x)}\,\to I^1}{I(x)\to(T(x)\to E(x))}\,\to I^2}{\forall x(I(x)\to(T(x)\to E(x)))}\,\forall I$$

Variable *x* does not occur free in any uncanceled assumption.

A Clarification: What does the Derivation Establish?

$$\dfrac{\dfrac{\dfrac{\dfrac{[I(x)]^2 \qquad \dfrac{\dfrac{[T(x)]^1}{[\forall y(I(y)\to E(y))]^3}}{I(x)\to E(x)}\,\forall E}{E(x)}\,\to E}{T(x)\to E(x)}\,\to I^1}{I(x)\to(T(x)\to E(x))}\,\to I^2}{\forall x(I(x)\to(T(x)\to E(x)))}\,\forall I$$

The derivation on this page is correct, but we should be clear about what it establishes.

It establishes that

$$\forall y(I(y)\to E(y)) \vdash \forall x(I(x)\to(T(x)\to E(x)))$$

It does ***not*** establish that

$$\vdash \forall x(I(x)\to(T(x)\to E(x)))$$

The derivation rests on the uncanceled assumption

$$\forall y(I(y)\to E(y))$$

And Now the Rules for the Existential Quantifier

Recall (2): Derivation Rules for the Existential Quantifier

$$\frac{\phi(t)}{\exists x\phi(x)}\exists I$$

$$\frac{\exists x\phi(x) \qquad \begin{array}{c}[\phi(x)]^i \\ \vdots \\ \psi\end{array}}{\psi}\exists E^i$$

<u>Restriction on ∃E:</u> Variable *x* cannot occur free in ψ and *x* cannot occur free in any assumptions in the sub-derivation of ψ except for $\phi(x)$.

$\neg\forall x\phi(x)$ *is equivalent to* $\exists x\neg\phi(x)$

$$\dfrac{\dfrac{\dfrac{[\neg\forall x\phi(x)]^3 \qquad \dfrac{\dfrac{\dfrac{[\neg\exists x\neg\phi(x)]^2 \qquad \dfrac{[\neg\phi(x)]^1}{\exists x\neg\phi(x)}\,\exists I}{\bot}\,{\to}E}{\phi(x)}\,RAA^1}{\forall x\phi(x)}\,\forall I}{\bot}\,{\to}E}{\exists x\neg\phi(x)}\,RAA^2}{\neg\forall x\phi(x) \to \exists x\neg\phi(x)}\,{\to}I^3$$

$$\dfrac{\dfrac{\dfrac{[\exists x\neg\phi(x)]^3 \qquad \dfrac{[\neg\phi(x)]^1 \qquad \dfrac{[\forall x\phi(x)]^2}{\phi(x)}\,\forall E}{\bot}\,{\to}E}{\bot}\,\exists E^1}{\neg\forall x\phi(x)}\,{\to}I^2}{\exists x\neg\phi(x) \to \neg\forall x\phi(x)}\,{\to}I^3$$

The Transformative Power of Negation (1)

For any formula ϕ, the following hold:

$\forall x \neg \phi(x)$ *is equivalent to* $\neg \exists x \phi(x)$

$\neg \forall x \phi(x)$ *is equivalent to* $\exists x \neg \phi(x)$

You should represent this pictorially. When negation moves from the inside to the outside of a quantifier, or from the outside to the inside of a quantifier, the negation changes the quantifier. If the quantifier is universal, the passage of negation makes the quantifier existential. If the quantifier is existential, the passage of negation makes the quantifier universal.

The Transformative Power of Negation (2)

From (classical) propositional logic, we have that

$$\neg(\phi \wedge \psi) \quad \textit{is equivalent to} \quad \neg\phi \vee \neg\psi$$

$$\neg(\phi \vee \psi) \quad \textit{is equivalent to} \quad \neg\phi \wedge \neg\psi$$

$$\neg\neg\phi \quad \textit{is equivalent to} \quad \phi$$

You should represent this pictorially. When negation goes through a conjunction, it turns the conjunction into a disjunction and it negates each of the conjuncts (now turned disjuncts). Similarly, when negation goes through a disjunction, it turns the disjunction into a conjunction and it negates each of the disjuncts (now turned conjuncts)

The Power of Negation in Action

Example 1: $\forall x \forall y \exists z \neg \phi$

is equivalent to

$\neg \exists x \exists y \forall z \phi$

Example 2: $\exists x \exists y \forall z (\neg R(x, y) \vee \neg R(y, z))$

is equivalent to

$\exists x \exists y \forall z \neg (R(x, y) \wedge R(y, z))$

is equivalent to

$\neg \forall x \forall y \exists z (R(x, y) \wedge R(y, z))$

The Power of Negation at a Glance

For any formula ϕ, the following hold:

$\forall x \neg \phi(x)$ *is equivalent to* $\neg \exists x \phi(x)$

$\neg \forall x \phi(x)$ *is equivalent to* $\exists x \neg \phi(x)$

From (classical) propositional logic, we have that

$\neg(\phi \wedge \psi)$ *is equivalent to* $\neg \phi \vee \neg \psi$

$\neg(\phi \vee \psi)$ *is equivalent to* $\neg \phi \wedge \neg \psi$

$\neg\neg\phi$ *is equivalent to* ϕ

Remember: There is a connection between $\forall$ and $\wedge$ and a connection between $\exists$ and $\vee$

Derivability and Logical Consequence

Derivability in Predicate Logic: $\vdash$

$\vdash \psi$ *iff*

there is a derivation of ψ in which all assumptions are canceled

$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$ *iff*

there is a derivation of ψ from uncanceled assumptions $\phi_1, \phi_2, \ldots, \phi_k$

A derivation is a tree-like arrangement of formulas which obeys the derivation rules for propositional and predicate logic.

Validity and Logical Consequence

Validity:

$\vDash \psi$

iff

all models ***M*** make ψ true

Logical Consequence:

$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$

iff

all models ***M*** that make $\phi_1, \phi_2, \ldots, \phi_k$ true make also ψ true

*Logical consequence is a **if-then universally quantified** claim:*

$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$

iff

for all models ***M*** [if ***M*** makes $\phi_1, \phi_2, \ldots, \phi_k$ true,
then ***M*** makes ψ true, as well]

Syntactic Standpoint

$\vdash \psi$

iff

there is a derivation of ψ in which all assumptions are canceled

$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$

iff

there is a derivation of ψ from uncanceled assumptions $\phi_1, \phi_2, \ldots, \phi_k$

Semantic Standpoint

$\vDash \psi$

iff

all models $\boldsymbol{M}$ make ψ true

$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$

iff

all models $\boldsymbol{M}$ which make $\phi_1, \phi_2, \ldots, \phi_k$ true make also ψ true

Finite *versus* Infinite Tasks

$\vdash$ **or** $\vDash$	***How to Establish the Claim?***	***Finite or Infinite Task?***
$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$	construct **one** derivation of ψ from uncanceled assumptions $\phi_1, \phi_2, \ldots, \phi_k$	*Finite*
$\vdash \psi$	construct **one** derivation of ψ in which all assumptions are canceled	*Finite*
$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$	consider **all** models that makes true $\phi_1, \phi_2, \ldots, \phi_k$ and check whether all such models make true ψ as well	*Infinite*
$\vDash \psi$	consider **all** models and check whether they all make true ψ	*Infinite*

$\nvdash$ or $\nvDash$	***How to Establish the Claim?***	***Finite or Infinite Task?***
$\phi_1, \phi_2, \ldots, \phi_k \nvdash \psi$	consider **all** derivations and check that no one establishes ψ from uncanceled assumptions $\phi_1, \phi_2, \ldots, \phi_k$	*Infinite*
$\nvdash \psi$	consider **all** derivations and check that no one establishes ψ	*Infinite*
$\phi_1, \phi_2, \ldots, \phi_k \nvDash \psi$	construct **one** model that makes true $\phi_1, \phi_2, \ldots, \phi_k$ and that does **not** make true ψ	*Finite*
$\nvDash \psi$	construct **one** model that does **not** make true ψ	*Finite*

Soundness and Completeness

The Equivalence of $\vdash$ and $\vDash$ in Predicate Logic

$$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$$

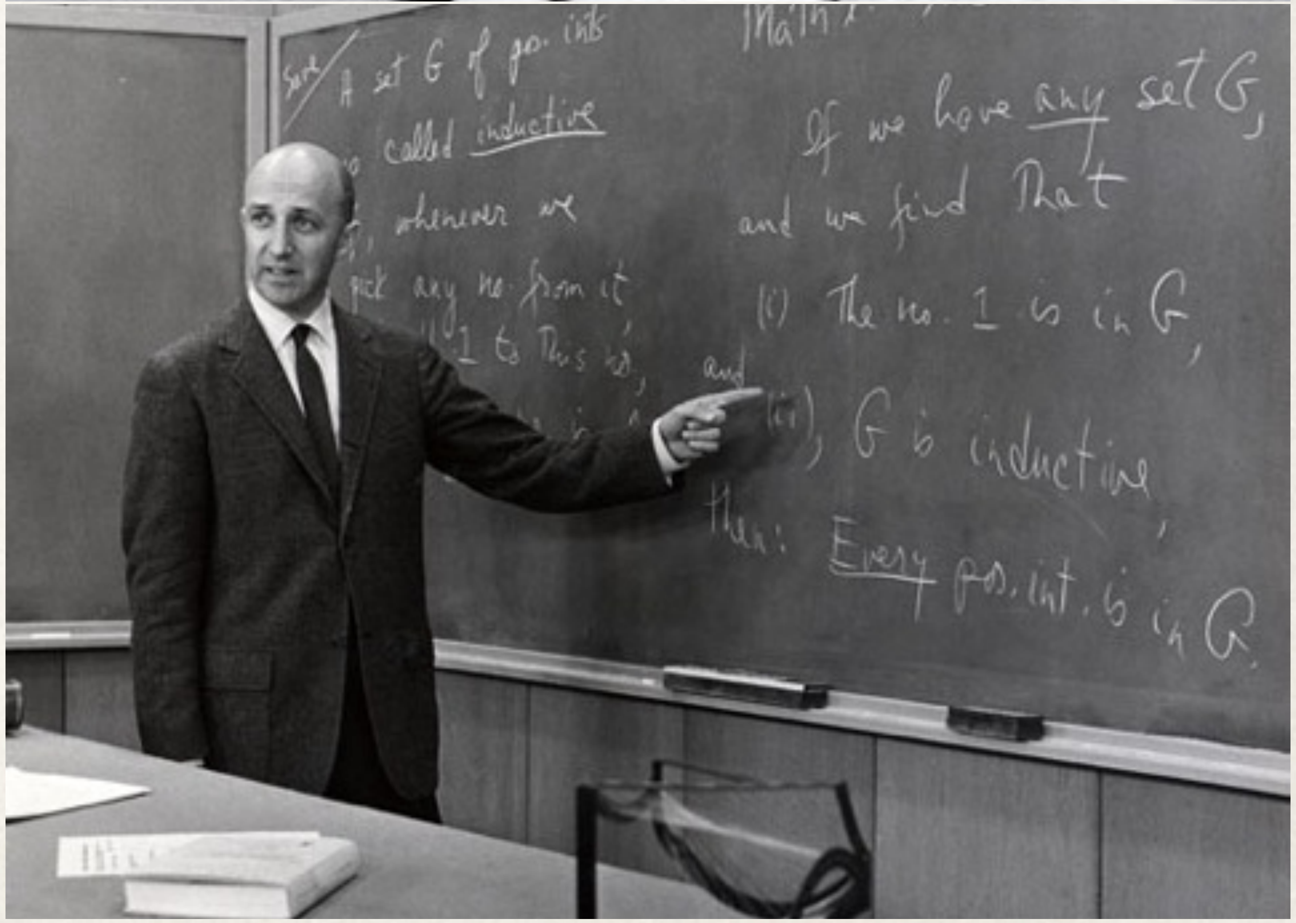
SOUNDNESS $\Searrow$ $\Nwarrow$ **COMPLETENESS**

$$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$$

You will have to take a more advanced logic course (e.g. **PHIL 151***) to see how the proof goes.*

The completeness of Predicate Logic was proven by Gödel in 1929

Leon Henkin from UC, Berkeley simplified the proof of completeness in 1947



$\phi_1, \phi_2, \ldots, \phi_k \vdash \psi$	construct **one** derivation of ψ from uncanceled assumptions $\phi_1, \phi_2, \ldots, \phi_k$	*Syntactic Standpoint* *Finite Task*
Completeness $\Uparrow$		$\Downarrow$ Soundness
$\phi_1, \phi_2, \ldots, \phi_k \vDash \psi$	consider **all** models that makes true $\phi_1, \phi_2, \ldots, \phi_k$ and check whether all such models make true ψ	*Semantic Standpont* *Infinite Task*