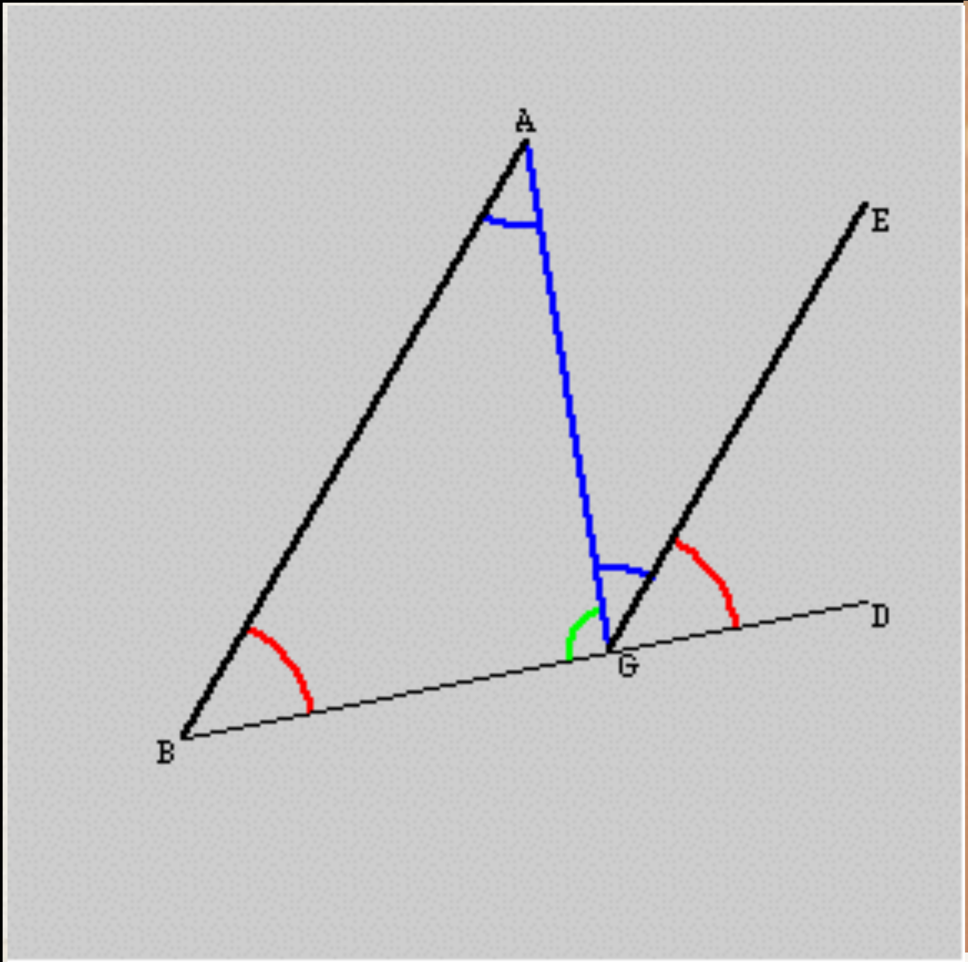
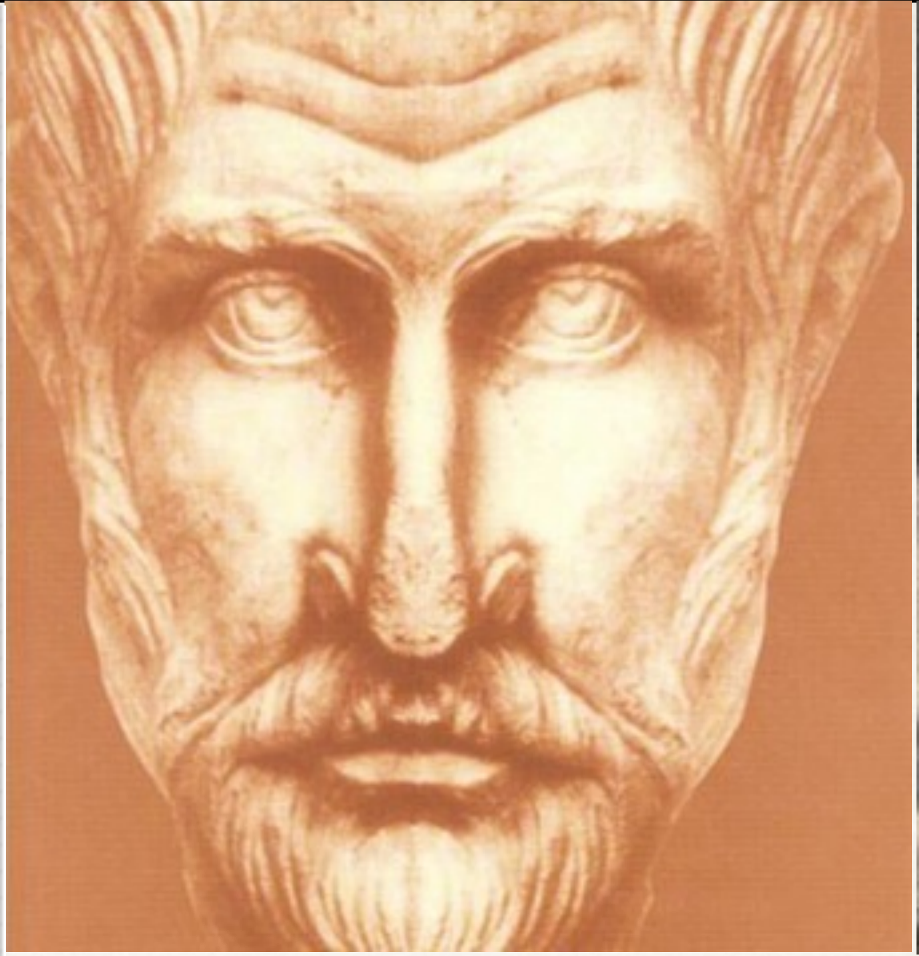


Euclid's proposition I.32



Proclus

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PHIL 50 - Introduction to Logic

Marcello Di Bello, Stanford University, Spring 2014

Week 8 — Monday Class - Validity and Derivations in Predicate Logic

Recall (1) — The Notion of a Model

A model ***M*** consists of

- a domain ***D*** of objects;
- an interpretation function ***I*** which
    - assigns constant symbols to objects
    - assigns 1-place predicate symbols to sets of objects
    - assigns 2-place predicate symbols to sets of pairs of objects
- an assignment ***g*** which assigns variable symbols to objects

More informally, we could say that a model describes

- which objects/individuals exist in a domain of discourse
- how words connect up to those objects/individuals

Recall (2) — Truth in a Model: $\langle D, I, g\rangle \vDash \phi$ (or equivalently $M \vDash \phi$)

$\langle D, I, g\rangle \vDash P(c)$ *iff* $I(c) \in I(P)$

$\langle D, I, g\rangle \vDash R(c_1, c_2)$ *iff* $\langle I(c_1), (I(c_2)\rangle \in I(R)$

$\langle D, I, g\rangle \vDash \neg \phi$ *iff* $\langle D, I, g\rangle \nvDash \phi$

$\langle D, I, g\rangle \vDash \phi \wedge \psi$ *iff* $\langle D, I, g\rangle \vDash \phi$ *and* $\langle D, I, g\rangle \vDash \psi$

$\langle D, I, g\rangle \vDash \phi \vee \psi$ *iff* $\langle D, I, g\rangle \vDash \phi$ *or* $\langle D, I, g\rangle \vDash \psi$

$\langle D, I, g\rangle \vDash \phi \rightarrow \psi$ *iff* $\langle D, I, g\rangle \vDash \phi$ *implies* $\langle D, I, g\rangle \vDash \psi$

$\langle D, I, g\rangle \vDash \exists x\phi$ *iff* there is a d [$d \in D$ and $\langle D, I, g_{[x:=d]}\rangle \vDash \phi$]

$\langle D, I, g\rangle \vDash \forall x\phi$ *iff* for all d [*if* $d \in D$, *then* $\langle D, I, g_{[x:=d]}\rangle \vDash \phi$]

A Remark about Truth in a Model

Consider the simplest case

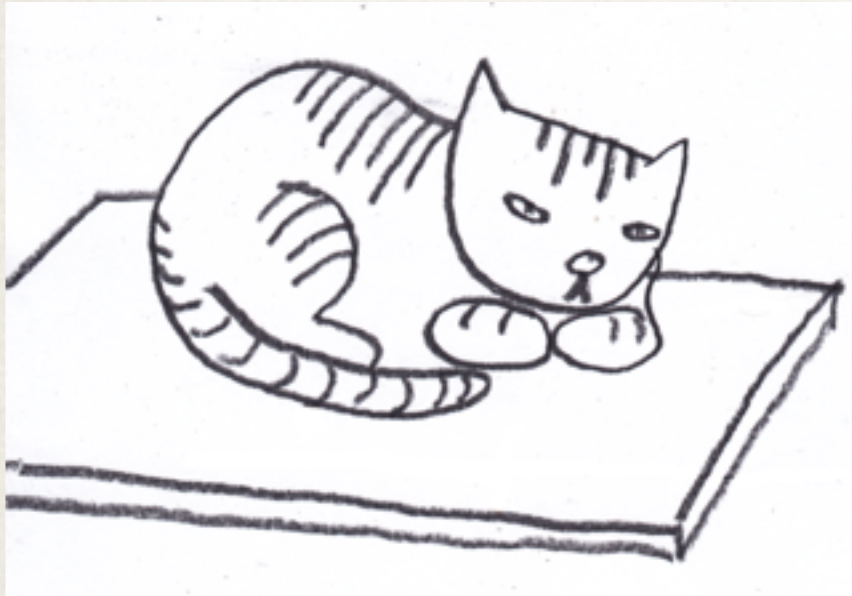
$\langle \boldsymbol{D, I, g} \rangle \vDash \mathbf{P}(\boldsymbol{c})$ *iff* $\mathbf{I(c)} \in \mathbf{I(P)}$

What the above *iff*-claim says is that

the formula **P(c)** is true in model $\langle$**D, I, g**$\rangle$ *iff*
the object corresponding to ***c*** belongs to the set corresponding to ***P***

Is it a model-independent fact whether the object corresponding to ***c*** belongs to the set corresponding to ***P*? No.** It depends on how we interpret the linguistic items ***c*** and ***P***. So, **truth** here has to do with **the relation between language and domain *D***. *No language, no truth.*

The Importance of Finding out What Words Mean



Intuitively, we would say that ***"the cat is on the mat"*** is TRUE relative to the situation depicted above.

Intuitively, we would say that ***"the cat is on the mat"*** is FALSE relative to the situation depicted above.

The notion of *truth in a model*, however, behaves differently. We first need to define what "***cat***", "***mat***" and "***on***" refer to, and once we have done so, we can check whether "**the cat is on the mat**" is true (relative to our model) or not.

According to the notion of truth in a model, the truth of a sentence crucially depends on the meaning we assign to the words we use. More generally, we can say that ***finding the truth is not only about finding out what the world is like, but mostly about finding out what words mean***.

From Truth in a Model.........
.........to Truth in **ALL** Models

Two Semantic Notions: Truth in a Model *and* Validity

Truth in a model

$M \vDash \phi$ *iff* **model** M **makes** ϕ **true, where** $\mathbf{M}=\langle D, I, g\rangle$

Validity

$\vDash \phi$ *iff* **for all models** M**, it holds that** $M \vDash \phi$

Example of a Valid Formula

Claim: $\vDash \forall x(P(x) \vee \neg P(x))$

Proof of the claim: Consider an arbitrary model M. We want to show that $M \vDash \forall x(P(x) \vee \neg P(x))$. By definition we have that

$\langle D, I, g\rangle \vDash \forall x(P(x) \vee \neg P(x))$

iff for all **d**, if $d \in D$, then $\langle D, I, g_{[x:=d]}\rangle \vDash P(x) \vee \neg P(x)$

iff for all **d**, if $d \in D$, then $\langle D, I, g_{[x:=d]}\rangle \vDash P(x)$ or $\langle D, I, g_{[x:=d]}\rangle \vDash \neg P(x)$

iff for all **d**, if $d \in D$, then $\langle D, I, g_{[x:=d]}\rangle \vDash P(x)$ or $\langle D, I, g_{[x:=d]}\rangle \nvDash P(x)$

iff for all **d**, if $d \in D$, then $g_{[x:=d]}(x) \in I(P)$ or $g_{[x:=d]}(x) \notin I(P)$

iff for all **d**, if $d \in D$, then $d \in I(P)$ or $d \notin I(P)$

It holds that [for all **d**, if $d \in D$, then $d \in I(P)$ or $d \notin I(P)$], because every object is either in a (*non-fuzzy*) set or it is not. So, $\mathbf{M} \vDash \forall \mathbf{x}(\mathbf{P}(\mathbf{x}) \vee \neg \mathbf{P}(\mathbf{x}))$. Since M was arbitrary, it follows that for all M, $M \vDash \forall x(P(x) \vee \neg P(x))$.

Hence, $\vDash \forall x(P(x) \vee \neg P(x))$.

The proof that a formula is valid often rests on using an arbitrary model M. You might be puzzled by the use of "arbitrary" items in your proofs, and you might wonder what such things look like. ***What is an arbitrary object, after all?*** Hopefully, these doubts will be (partly) cleared up today as we discuss the rule of universal introduction.

Example of an Invalid Formula

Claim: $\not\models \forall x(P(x)) \vee \forall x \neg (P(x))$

Proof of the claim: To show that a formula is not valid, it suffices to construct one model M which makes the formula false. To this end, let **D={□, △}** and let **I(P)={□}**. We should check that for M thus defined, $M \not\models \forall x(P(x)) \vee \forall x \neg (P(x))$. By definition we have that

$$M \models \forall x(P(x)) \vee \forall x \neg ((P(x))$$

iff

[for all **d**, if $d \in D$, then $d \in I(P)$] **or** [for all **d′**, if $d' \in D$, then $d' \notin I(P)$] (*)

However, we have defined our model M such that

[there is a **d**, $d \in D$ and $d \notin I(P)$] **and** [there is a **d′**, $d' \in D$ and $d' \in I(P)$] (**)

Now, (**) is the negation of (*), so $M \not\models \forall x(P(x)) \vee \forall x \neg (P(x))$.

Hence, $\not\models \forall x(P(x)) \vee \forall x \neg (P(x))$.

Mind the Difference…

$\models \forall x(P(x) \vee \neg P(x))$

$\not\models \forall x(P(x)) \vee \forall x \neg (P(x))$

It is always true that everybody is either American or is not American.

It is **NOT** always true that either everybody is American or everybody is not American.

While the difference might not be completely clear in natural language sentences, it is very clear in predicate logic.

Establishing Validity and Invalidity

To show that a formula is **valid**, one should show that the formula is **true in all possible models**. Since one cannot go through all possible models (which could be infinite in number), the only method here is to consider an **arbitrary model**.

Consider all models or an arbitrary model

To show that a formula is **invalid**, one should only show that there is **one model *M* which makes the formula false**. To this end, one should construct a model that falsifies the formula by defining a **domain *D*** and an **interpretation function *I***.

Construct one (counter)model

Two Standpoints

Semantic Standpoint:
Truth and validity

Syntactic Standpoint:
Derivability

From Validity to Derivability

(Semantic) Validity

$\vDash \phi$ *iff* **for all models** M**, it holds that** $M \vDash \phi$

(Syntactic) Derivability

$\vdash \phi$ *iff* **there is a derivation of** ϕ **from no assumptions**

You are familiar with the the symbols $\vDash$ and $\vdash$ from propositional logic. The novelty now is that we are dealing with the validity and derivability of formulas of ***predicate logic***, and not simply of formulas of propositional logic.

Propositional logic:

Validity

$\vDash \phi$ *iff* **for all valuations *V*, it holds that $V \vDash \phi$**

Derivability

$\vdash \phi$ *iff* there is a derivation of ϕ from no assumptions using the **rules of derivation for propositional logic.**

Predicate logic:

Validity

$\vDash \phi$ *iff* **for all models *M*, it holds that $M \vDash \phi$**

Derivability

$\vdash \phi$ *iff* there is a derivation of ϕ from no assumptions using the **rules of derivation for predicate logic.**

Derivation Rules for Predicate Logic

The derivation rules for predicate logic **include** the derivation rules for propositional logic (recall week 3 of the course)

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The derivation rules for predicate logic include **—in addition—**four derivation rules that are specific to the existential and the universal quantifier

Some Preliminary Notions (1)
Free *versus* Bound Variables

A variable x occurs **free** in a formula whenever x **does not occur within the scope of the quantifier** $\forall x$ **or** $\exists x$.

Examples:

x occurs free in $P(x)$

both x and y occur free in $R(x, y)$

x occurs free in $\forall y(R(x, y))$ although y does not occur free

A variable x occurs **bound** in a formula whenever x **does occur within the scope of the quantifier** $\forall x$ **or** $\exists x$.

Examples:

x occurs bound in $\forall x(P(x))$

both x and y occur bound in $\forall x(\forall y(R(x, y)))$

y occurs bound in $\forall y(R(x, y))$ although x occurs free

Some Preliminary Notions (2)
The Scope of a Quantifier

The **scope of a quantifier** is the (sub)formula which begins with the open bracket `**(**' that immediately follows the quantifier and which ends when the bracket is closed by `**)**'.

Examples:

the scope of $\forall x$ in $\forall x(P(x))$ is the formula $P(x)$

the scope of $\forall x$ in $\forall x(\forall y(R(x, y)))$ is the formula $\forall y(R(x, y))$

the scope of $\forall y$ in $\forall y(R(x, y))$ is the formula $R(x, y)$

the scope of $\forall x$ in $\forall x(R(x, y)) \wedge P(x)$ is the formula $R(x, y)$

So Let's See the Derivation Rules That Are Specific to Predicate Logic

Derivation Rules for the Universal Quantifier

Conventions. (a) Let $\phi(x)$ be a placeholder for a formula of predicate logic of arbitrary complexity where x occurs free in ϕ. (b) Let $\phi(t)$ be the placeholder for a formula of predicate logic of arbitrary complexity, where t is a placeholder for a variable symbol or a constant symbol.

$$\frac{\forall x \phi(x)}{\phi(t)} \forall E$$

$$\frac{\phi(x)}{\forall x \phi(x)} \forall I$$

<u>Restriction on ∀I</u>

*Variable **x** cannot occur free in any uncanceled assumption on which* $\phi(x)$ *depends.*

The Rule of Universal Elimination

$$\frac{\forall x \phi(x)}{\phi(t)}\ \forall E$$

***Illustration*:** Suppose I have a derivation for the claim that every politician is corrupt, *i.e.* $\forall x(P(x)\rightarrow C(x))$. Rule $\forall E$ allows us to derive a claim about a specific individual, e.g. ***obama*.**

$$\frac{\forall x(P(x) \rightarrow C(x))}{P(obama) \rightarrow C(obama)}\ \forall E$$

Here is an example of a larger derivation using rule **∀E**

$$\dfrac{\dfrac{[\forall x(P(x) \rightarrow C(x))]^*}{P(obama) \rightarrow C(obama)}\ \forall E \qquad [P(obama)]^*}{C(obama)}\ \rightarrow E$$

NB: The assumptions marked by * are uncanceled.

And Now the Rule of Universal Introduction

$$\frac{\phi(x)}{\forall x \phi(x)}\ \forall\mathbf{I}$$

Restriction on ∀I

*Variable **x** cannot occur free in any uncanceled assumption on which **φ(x)** depends.*

The rough idea behind rule **∀I** and its associated restriction is that once we manage to show that

x is an φ <u>for some arbitrary</u> *x,*

then we have managed to show that

x is φ <u>for all *x*</u>

More will be said on the importance and role of the restriction in the slides to follow.

Illustration of the use of ∀I

Let **x** be an odd number. So, **x=2k+1**, with **k** some natural number.
Now, we want to show that $\mathbf{x^2}$ is also odd**.** We have:

$$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2+4k+1 \\ &= 2(2k^2+2k)^2+1 \end{aligned}$$

Sine it can be written in this form, $\mathbf{x^2}$ is odd. So, if **x** is odd, $\mathbf{x^2}$ is also odd. Further, **x** was <u>arbitrary</u>, so <u>for every</u> odd number **x**, also $\mathbf{x^2}$ is odd.

$$\dfrac{\mathbf{Odd(x) \rightarrow Odd(x^2)}}{\mathbf{\forall x(Odd(x) \rightarrow Odd(x^2))}}\ \forall\mathbf{I}$$

This piece of reasoning is very powerful because it allows us to establish **a claim that applies to <u>all</u> odd numbers** just by **reasoning about <u>one</u> (arbitrary) odd number**.

The Universal Generalization Problem

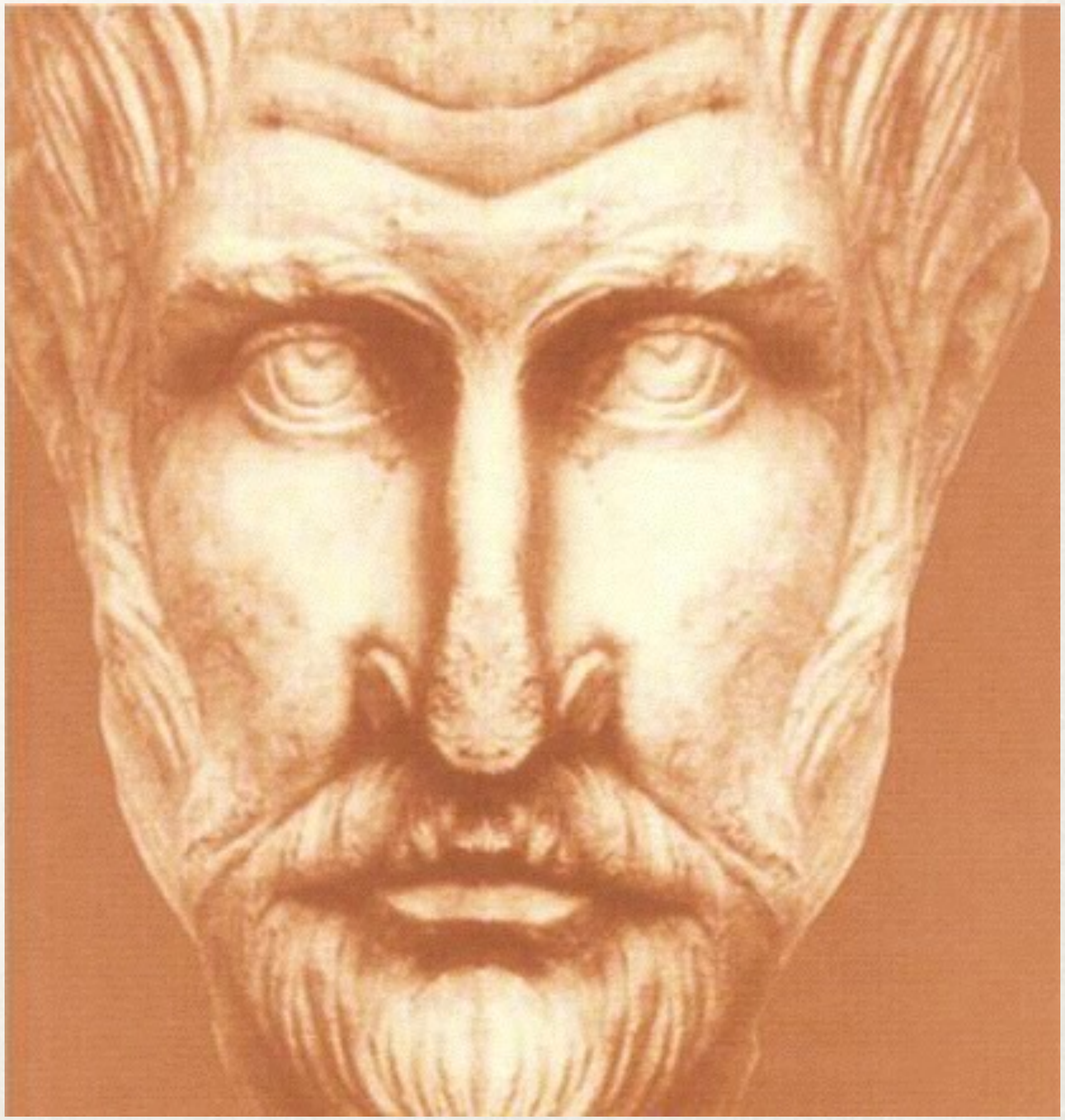
Rule **∀I** allows us to derive a universal claim of the form $\boldsymbol{\forall x \phi(x)}$ from a particular, non-universal claim particular of the form $\boldsymbol{\phi(x)}$.

$$\frac{\phi(x)}{\forall x \phi(x)}\ \forall\mathbf{I}$$

But why are we entitled to make this reasoning step? How can a claim about **one** object turn into a claim about **all** objects? This is what we might call the *universal generalization problem*.

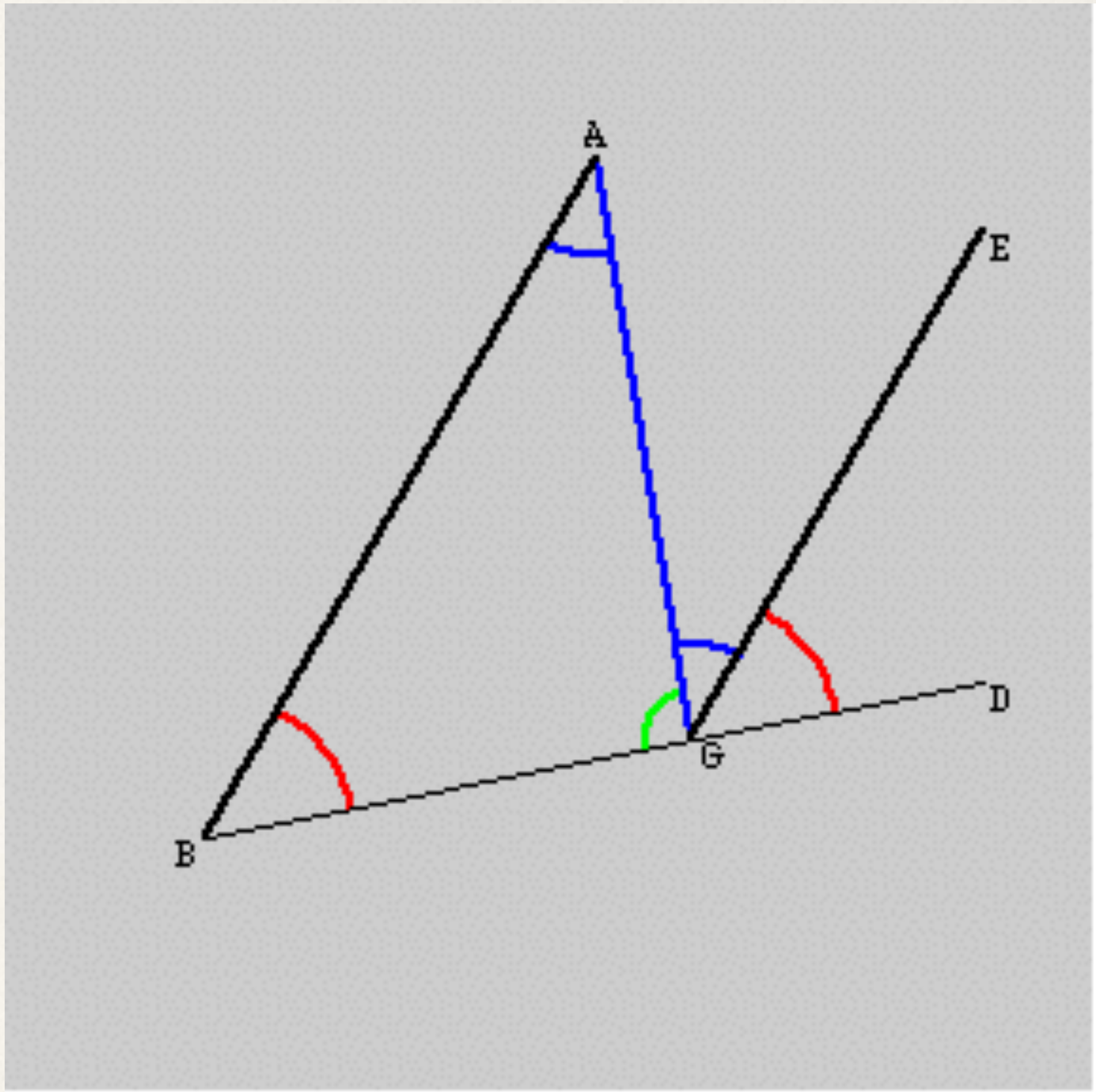
The Universal Generalization Problem Stated by the Neoplatonic Philosopher Proclus

"Mathematicians are used to draw what is in a way a double conclusion: in fact, when they have shown something to hold of the given figure, they infer that it holds in general, going from the particular conclusion to the general one" (Proclus 412-485)



Euclid's Proposition I.32

Proposition I.32. *In **ANY** triangle, the sum of the three internal angles of the triangle is equal to two right-angles.*



Euclid establishes this proposition by reasoning about a (*particular*) triangle ***ABC***. He shows that ***ABC***'s internal angles equal two right angles. Then, he draws the general conclusion that—*for all triangles*—the internal angles are equal to two right angles.

What is the justification for the last reasoning step, i.e. the generalization to all triangles?

The Intuitive Answer to the Universal Generalization Problem

Establishing a conclusion about **one particular object** of a certain type—e.g. Euclid's conclusion about *triangle ABC*—entitles one to draw a conclusion about **all objects** of a certain type—e.g. *all triangles*—insofar as the reasoning was about an **arbitrary object** of a certain type—e.g. Euclid's reasoning was about *an arbitrary triangle*.

PROBLEM: What is an arbitrary object of a certain type? For instance, what is an arbitrary triangle? Are there such things as arbitrary objects of a certain type? Are they abstractions or what?

The Formal Answer to the Universal Generalization Problem

$$\frac{\phi(x)}{\forall x \phi(x)}\ \forall\mathbf{I}$$

Restriction on ∀I

*Variable **x** cannot occur free in any uncanceled assumption on which **φ(x)** depends.*

The intuitive requirement that *x* be arbitrary is formally encoded by the restriction that *x* cannot occur free in any uncanceled assumptions on which **φ(x)** depends. If *x* were to occur free in some uncanceled assumption, this would mean that *x* was not arbitrary after all, but that additional assumptions about the nature of *x* had been made.