Uhm…

Gerard Gentzen

PHIL 50 - Introduction to Logic

Marcello Di Bello, Stanford University, Spring 2014

Week 8 — Wednesday Class - Derivations in Predicate Logic and Identity

Recall (1): Derivation Rules for the Universal Quantifier

Conventions. (a) Let $\phi(x)$ be a placeholder for a formula of predicate logic of arbitrary complexity where x occurs free in ϕ. (b) Let $\phi(t)$ be the placeholder for a formula of predicate logic of arbitrary complexity, where t is a placeholder for a variable symbol or a constant symbol.

$$\frac{\forall x \phi(x)}{\phi(t)}\ \forall E$$

$$\frac{\phi(x)}{\forall x \phi(x)}\ \forall I$$

<u>Restriction on ∀I</u>

Variable x cannot occur free in any uncanceled assumption on which $\phi(x)$ depends.

Recall (1): The restriction on the Universal Introduction Rule

$$\frac{\phi(x)}{\forall x \phi(x)}\ \forall\mathbf{I}$$

<u>Restriction on **∀I**</u>

*Variable **x** cannot occur free in any uncanceled assumption on which **$\phi(x)$** depends.*

The restriction on rule **∀I** amounts to the requirement that *x* be **<u>arbitrary</u>**. This requirement is formally encoded by the restriction that ***x*** **cannot occur free in any uncanceled assumption on which *$\phi(x)$* depends**. For if *x* were to occur free in some uncanceled assumption, this would mean that *x* was not arbitrary after all, but that additional assumptions about the nature of *x* had been made.

Derivation Rules for the Existential Quantifier

$$\frac{\phi(t)}{\exists x\phi(x)}\ \exists I$$

$$\frac{\exists x\phi(x) \qquad \begin{matrix}[\phi(x)]^i \\ \vdots \\ \psi\end{matrix}}{\psi}\ \exists E^i$$

<u>Restriction on **∃E**:</u> Variable *x* cannot occur free in ***ψ*** and ***x*** cannot occur free in any assumptions in the sub-derivation of ***ψ*** except for ***ϕ(x)***.

Existential Introduction

Illustration of Existential Introduction

$$\frac{\phi(t)}{\exists x \phi(x)}\exists I$$

If you derive that ϕ holds for some specific t, then you can also derive that there is a generic x for which ϕ holds.

$$\frac{\mathit{Mountain(half\text{-}dome)}}{\exists x \mathit{Mountain}(x)}\exists I$$

If you derive that Half Dome is a mountain, then you can also derive that there is a mountain.

What About Creatures of Fiction?

If you establish that Sherlock Holmes is infallible, does it follow that there is someone infallible? That's strange. Holmes is a creature of fiction and infallible people might not exist for real.

$$\frac{\mathit{Infallible}(sh)}{\exists x \mathit{Infallible}(x)}\ \exists\mathrm{I}$$

BUT: The claim that **∃xϕ(x)** means that there exists an object *x* in your domain such that **x** is **ϕ**. This does not mean that the object in question exists in reality. All it's been established is that the object exists in your domain, where ***the domain can be imaginary or real***.

A Clarification

Existentially quantified formula

$\exists x P(x)$

Natural language formulations:

Someone is ***P***
At least one object is ***P***

Quasi formalizations:

There is an **x** that is ***P***
There exists an **x** that is ***P***

Existential Elimination

What's the Point of Existential Elimination

$$\dfrac{\exists x\phi(x) \qquad \begin{matrix}[\phi(x)]^i \\ \cdot \\ \cdot \\ \psi\end{matrix}}{\psi}\ \exists E^i$$

Restriction on $\exists$E:

Variable x cannot occur free in ψ and x cannot occur free in any assumptions in the sub-derivation of ψ except for $\phi(x)$.

Suppose you know that there exists an expert skier, i.e. $\exists x S(x)$.

What can you derive from $\exists x S(x)$?

Rules $\exists E$ allows you to derive conclusions from existentially quantified claims. How?

Illustration of Good Reasoning Involving Existential Elimination

Let's say you know that

(a) someone is an expert skier; and

(b) every expert skier can ski down a black trail;

Now, it seems right to conclude that

(c) someone can ski down a black trail

(a) $\exists x S(x)$

(b) $\forall x (S(x) \rightarrow B(x))$

.

.

(c) $\exists x B(x)$

We can represent this reasoning as a derivation in predicate logic using $\exists E$

Squaw Valley for you….

The Reasoning Using Rule $\exists E$

(a) $\exists x S(x)$
(b) $\forall x(S(x) \to B(x))$
(c) $\exists x B(x)$

$$\dfrac{\exists x S(x) \qquad \dfrac{\dfrac{[S(x)]^1 \qquad \dfrac{\forall x(S(x) \to B(x))}{S(x) \to B(x)}\,\forall E}{B(x)}\,\to E}{\exists x B(x)}\,\exists I}{\exists x B(x)}\,\exists E^1$$

Illustration of Bad Reasoning Involving Existential Elimination

Let's say you know that

(a) someone is an expert skier;

(b*) if x is an expert skier, x wears a tuxedo while skiing;

Now, it is wrong to conclude from (a) and (b*) alone that

(c*) someone wears a tuxedo while skiing.

(a) $\exists x S(x)$

(b*) $S(x) \rightarrow T(x)$

.

. ??

.

(c*) $\exists x T(x)$

Claim **(a)** does not specify any particular x who is the expert skier, while claim **(b*)** fixes on a particular x. This mismatch between **(a)** and **(b*)** makes the reasoning bad.

Generic x *versus* Specific x

(a) $\exists x S(x)$

(b*) $S(x) \rightarrow T(x)$

.

. ??

.

(c*) $\exists x T(x)$

The problem with the reasoning is that claim **(a)** does not specify any particular x who is the expert skier. Instead, claim**(b*)** fixes on an particular x who has the peculiar feature that if x is an expert skier, x wears a tuxedo while skying.

Do not be deceived by the fact that we are using x in booth cases. In the case of $\exists x S(x)$, we are simply saying that there is some x (you can call it y, z,) such that x is S. In the case of $S(x) \rightarrow T(x)$, there is no quantifier, so we are picking a specific x.

A Misapplication of Rule $\exists E$

(a) $\exists x S(x)$

(b) $S(x) \to T(x)$

(c) $\exists x T(x)$

$$\dfrac{\exists x T(x) \qquad \dfrac{\dfrac{[S(x)]^1 \qquad S(x) \to T(x)}{T(x)}\,{\to}E}{\exists x T(x)}\,\exists I}{\exists x T(x)}\,E\exists^1 \textbf{\textit{ wrong!}}$$

The restriction that $\boldsymbol{x}$ *should not occur free in the subderivation of* $\boldsymbol{\exists x T(x)}$ *except for* $\boldsymbol{S(x)}$ *is violated.*

Looking at the Problem Semantically
The Reasoning (a), (b*), (c*) is Invalid

(a) $\exists x S(x)$

(b*) $S(x) \rightarrow T(x)$

.

. ???

(c*) $\exists x T(x)$

Consider a model *M* such that:

D={◝, ❡}

I(S)={◝} **I(T)=∅**

g(x)=❡

You can check that *M* makes true (a) because there is an element, namely ◝, which is **S.**

Further, since **g(x)** is interpreted as ❡, (b*) is true vacuously. The antecedent is false because $g(x) \notin I(S)$. So (b*) is true in *M*.

But (c*) is false in *M* because **I(T)** is empty.

Identity $=$

So far our language lacked a symbol for identity. Let's now introduce a symbol for identity.

What Does = Mean?

$\langle D, I, g\rangle \vDash (c_1 = c_2)$ *iff* $\langle \mathbf{I}(\mathbf{c}_1), \mathbf{I}(\mathbf{c}_2) \rangle \in \mathbf{I}(=)$

$\langle D, I, g\rangle \vDash (\mathbf{x} = y)$ *iff* $\langle \mathbf{g}(\mathbf{x}), \mathbf{g}(\mathbf{y}) \rangle \in \mathbf{I}(=)$

I(=) is a set of pairs because "=" is a *two-place predicate* after all. What's peculiar about **I(=)** is that each pair in the set must consist of the same object twice.

Illustration:

***I*(=) = { …**

⟨  ,  ⟩ , ⟨  ,  ⟩

… }

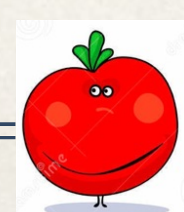
Illustration

$\langle D, I, g\rangle \vDash (c_1 = c_2)$ *iff* $\langle \mathrm{I}(c_1), \mathrm{I}(c_2) \rangle \in \mathrm{I}(=)$

Let ***M*** be as follows:

"b"

"a"

I(a)=  I(b)= 

a=a is true in M

b=b is true in M

a=b is true in M